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Holger Kammeyer

Introduction to Algebraic Topology

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Introduction to Algebraic Topology

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Preface

This book introduces the reader to the two most fundamental concepts of algebraic topology: the fundamental group and homology. We shall take a modern viewpoint so that we begin the course by studying basic notions from category theory. The fundamental group is afterwards treated as a special case of the fundamental groupoid. Accordingly, we first prove van Kampen's theorem in a categorical version due to R. Brown and then explain how to actually compute the fundamental group of an attaching space. We move on to present homology. To convey the idea, we construct simplicial homology and motivate the Eilenberg–Steenrod axioms of a homology theory. Next, we construct singular homology and show that it satisfies the axioms. Afterwards, we develop machinery for computing homology theories, give example calculations, and see some applications such as the Brouwer fixed point theorem and the Borsuk–Ulam theorem. Finally, we introduce cellular homology for CW complexes. As the concluding result, we show that the Eilenberg Steenrod axioms determine ordinary homology on CW complexes.

We assume the reader has taken an introductory course on topology and is familiar with point set topological concepts, the definition of the fundamental group, and covering theory. Such a course will certainly have covered the quotient topology, but since this concept is of fundamental importance for our purposes, we have decided to include a recap in Appendix A. Elementary algebra will likewise be applied throughout the text. In particular, we will work with modules over commutative rings and on rare occasions also with their tensor products. Each chapter ends with a couple of exercises. Some of them are not only meant to provide a test ground for working with the new concepts but they also establish additional facts and terminology which are helpful to know in the given context.

A myriad of excellent books on algebraic topology are available in the market. Some texts, for example, [29], choose a formal presentation and are well suited to continue one's curriculum with a course on homotopy theory. Others, like [8] are more geometrically minded and might be a better choice for subsequent specialization in low-dimensional topology. What this text tries to accomplish is to neither shy away from abstract concepts nor from providing geometric intuition or doing easy calculations, and at the same time do justice to the series and be a compact textbook: We present only as much material as we found reasonable to cover in a first semester graduate course on algebraic topology. The six chapters are divided into five sections each, so that in a typical 15-week semester with two meetings a week, one should cover one section per lecture on average. This is however a rough estimate as some sections are more substantial than others so that the blackboard presentation will need some shortcuts. However, we urge the lecturer not to try and skip the technical appearing Sect. 2.3 on cofibrations and homotopy pushouts. It secretly provides as much homotopy theory as we deem necessary for the correct presentation of the results in the ensuing chapters.

As another remark to the lecturer, let me point out a well-known dilemma when aiming for the uniqueness theorem of ordinary homology. At some moment, one will have to know that $\pi_n(S^n) \cong \mathbb{Z}$ or more precisely that $[S^n, S^n] \cong \mathbb{Z}$ meaning homotopy classes of maps $S^n \to S^n$ are classified by degree. This theorem just has no quick and easy proof. It is interesting to see how other introductory texts on algebraic topology circumvent this problem. For example, tom Dieck in his monograph [29] develops homotopy theory first, before introducing homology, so that the fact $\pi_n(S^n) \cong \mathbb{Z}$ is available once it is needed. Hatcher in [8] takes the more classic route of treating homology first and simply waits with the proof of the uniqueness theorem until after developing homotopy theory [8, Theorem 4.59]. Lück advances quickly to the uniqueness theorem [18, Satz 3.53] by taking the Freudenthal suspension theorem for granted. In this text, we suggest yet another road to resolve the issue and prove the simplicial approximation theorem as part of the chapter on simplicial homology which allows us to show $\pi_n(S^n) \cong \mathbb{Z}$ by a lemma given in [4, Lemma 11.13]. This has the virtue that the introduction of simplicial complexes serves more than only a didactic purpose.

The used background sources are as follows: Chaps. 1 and 2 loosely follow the presentation in [19, Chapter 2], though Chap. 1 gives a more extensive introduction to categorical concepts, and Sect. 2.3 incorporates material appearing in [4, 26, 29]. Chapter 3 is partly based on [8] with the section on simplicial approximation drawing from [20]. The material of Chap. 4 follows the default treatment and can for example be found in [8]. References for Chap. 5 are again [4] and [8], though some proofs are adapted to the more formal notion of cofibration instead of Hatcher's "good pairs." The main reference for the final Chap. 6 is [18], but some proofs have been revised considerably.

Additionally, I want to thank Roman Sauer for providing me with his handwritten lecture notes [21] which have served as an overall fundament of the course. I am moreover indebted to David Bückel for taking live LaTeXnotes when I first taught the material of this course so that I could thereafter simply extend his notes to the text at hand. Finally, I am grateful to Moritz Kerz for suggesting the inclusion of these notes into the Birkhäuser Compact Textbooks in Mathematics series. Without any one of these three, the book would not have come into being.

Düsseldorf, Germany Holger Kammeyer
December 5, 2021

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Contents

1 Basic Notions of Category Theory

Let us start the text with an easy question: Is the two dimensional sphere S^2 homeomorphic to the two dimensional torus $\mathbb{T}^2$? Both spaces are connected, compact 2-dimensional manifolds, and hence indistinguishable from the point set topological point of view. But there are compelling intuitive ideas why these spaces should be distinct: Consider a rubber band in $\mathbb{T}^2$ fixed at some point $x_0 \in \mathbb{T}^2$. If this rubber band is embedded in such a way that it winds once around the "hole" in the torus, as pictured on the right in Fig. 1.1, there seems to be no way whatsoever to continuously deform this band within $\mathbb{T}^2$ to the point x_0. In the sphere, however, it appears to be an easy task to contract a rubber band to a point, no matter how it is initially embedded.

Algebraic topology is the art of making these thoughts precise. If S^2 was homeomorphic to $\mathbb{T}^2$, then we would have $\pi_1(S^2, x_0) \cong \pi_1(\mathbb{T}^2, x_0)$. However, $\pi_1(S^2, x_0) = \{1\}$ and $\pi_1(\mathbb{T}^2, x_0) \cong \mathbb{Z} \times \mathbb{Z}$. So S^2 is not homeomorphic to $\mathbb{T}^2$. Here, the fundamental group defines a **functor** from the **category** of (pointed) topological spaces to the category of groups. To a large extent, algebraic topology is about constructing functors from categories of topological spaces to algebraic categories like groups, abelian groups, K-vector spaces, and R-modules.

Category theory provides vocabulary to formulate the transition of topological questions into algebraic problems in a precise and consistent manner. This is why it has long become the gold standard to develop algebraic topology in its terms. In this first chapter we present precisely as much of this language as we shall employ in the course.

1.1 Categories

In the first few semesters of studying math, one realizes that many constructions and arguments pop up repeatedly in different contexts. For instance, **products** are defined in virtually the same way, no matter whether we are dealing with products of

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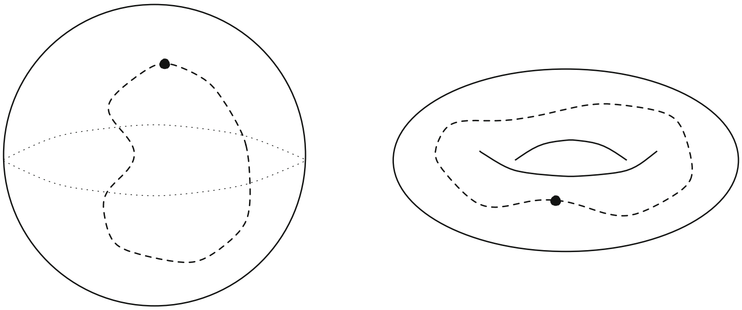


Fig. 1.1 Rubber bands embedded in S^2 and $\mathbb{T}^2$

groups, rings, or vector spaces. This raises the desire to explain the term "product" once and for all in an abstract fashion that would specialize to all the particular cases needed in mathematics. But to come up with a meaningful abstract definition of "$X \times Y$," it is indispensable to first convey in one way or another that "X" and "Y" should be two "instances" of the same "type"; or we had better say two **objects** in the same **category**.

Definition 1.1

A **category** $\mathcal{C}$ consists of a class of **objects** $\text{ob}(\mathcal{C})$ and a class of **morphisms** $\text{Hom}_{\mathcal{C}}(X, Y)$ associated with any two objects $X, Y \in \text{ob}(\mathcal{C})$. Morphisms are also called **arrows**

$$(f : X \to Y) \in \text{Hom}_{\mathcal{C}}(X, Y)$$

from the **domain** X to the **codomain** Y. They are subject to two conditions.

(i) Two morphisms can be **composed** if the codomain of the former is the domain of the latter. Given $f : X \to Y$ and $g : Y \to Z$, we obtain $g \circ f : X \to Z$ and composition is associative: for $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$, we have $h \circ (g \circ f) = (h \circ g) \circ f$.

(ii) For every object $X \in \text{ob}(\mathcal{C})$, there exists an **identity morphism** $\text{id}_X \in \text{Hom}_{\mathcal{C}}(X, X)$, such that for all $f : X \to A$ and $g : B \to X$ we have $f \circ \text{id}_X = f$ and $\text{id}_X \circ g = g$.

Remark 1.2

The reader might be confused by the word "class" in the definition. How come we cannot use the familiar term "set"? The reason is that also sets themselves are supposed to form a category, and talking about the "set of all sets" is logically troublesome. It admittedly looks shaky to try and overcome this issue by using a different word. But for our purposes, we merely think of "class" as a manner of speaking. Saying "Let X be from the class of sets" shall be equal in meaning to saying "Let X be a set"; a phrase that no one would ever complain about. A more foundational discussion would leave the realm of algebraic topology.

Table 1.1 Examples of categories and their objects and morphisms

Category	Objects	Morphisms	Isomorphisms
Set	Sets	Maps	Bijections
K-vect	K-vector spaces	K-linear maps	K-isomorphisms
R-mod	R-modules	R-linear maps	R-isomorphisms
Group	Groups	Group homomorphisms	Group isomorphisms
Ab	Abelian groups	Group homomorphisms	Group isomorphisms
Top	Topological spaces	Continuous maps	Homeomorphisms
$\mathsf{Top}_\bullet$	Pointed topological spaces	Continuous base point preserving maps	Base point preserving homeomorphisms
HoTop	Topological spaces	Homotopy classes of continuous maps	Homotopy classes of homotopy equivalences
$\mathsf{HoTop}_\bullet$	Pointed topological spaces	Pointed homotopy classes of continuous base point preserving maps	Pointed homotopy classes of base point preserving homotopy equivalences
$\mathsf{Top}^{(2)}$	Pairs (X, A) of a topological space X and a subspace A	Continuous maps that restrict to a map of the subspaces	Homeomorphisms that restrict to a homeomorphism of the subspaces

Morphisms whose domain coincides with the codomain are called **endomorphisms**. A morphism $f: X \to Y$ in a category $\mathcal{C}$ is called an **isomorphism** if there is a morphism $g: Y \to X$, called the **inverse** of f, such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. If f is an isomorphism and g_1, g_2 are two inverses, then

$$g_2 = g_2 \circ \mathrm{id}_Y = g_2 \circ (f \circ g_1) = (g_2 \circ f) \circ g_1 = \mathrm{id}_X \circ g_1 = g_1.$$

So inverses, if they exist, are unique. An endomorphism that is also an isomorphism is called an **automorphism**. Examples of categories abound. We gather the examples that are most relevant for algebraic topology in Table 1.1.

Let us make sure we understand the arrows in the **homotopy category** HoTop and the **pointed homotopy category** $\mathsf{HoTop}_\bullet$. We have

$$\mathrm{Hom}_{\mathsf{HoTop}}(X, Y) = \{f : X \to Y,\ f \text{ continuous}\}/\simeq$$

where the equivalence relation $f \simeq g$ is called **homotopy**. It means there exists a family of maps $H_t: X \to Y$ for $t \in I = [0, 1]$ defining a continuous map $H: X \times I \to Y$ such that $H_0 = f$ and $H_1 = g$. The composition is given by $\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$, $([f], [g]) \mapsto [g \circ f]$ where now "$\circ$" denotes the composition of maps (the composition in Set). One easily checks this is well-defined. Any representative $f: X \to Y$ of an isomorphism $[f]$ in HoTop is called a **homotopy equivalence**. So a homotopy equivalence $f: X \to Y$ has a **homotopy inverse**, meaning a continuous map $g: Y \to X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. In the special case that f is the inclusion of a subspace and that

g is a **retraction**, meaning $g \circ f = \mathrm{id}_X$ holds on the nose, the subspace X is called a **deformation retract** of Y and the homotopy H from $f \circ g$ to id_Y is called a **deformation retraction**. If H fixes the subspace X pointwise throughout, then H is called a **strong deformation retraction** and X is called a **strong deformation retract** of Y. A space is called **contractible** if it contains one of its points as a deformation retract (and is in particular nonempty). Equivalently, a space is contractible if it is homotopy equivalent to a one point space "•". There exists a contractible space that does not strongly deformation retract onto any of its points [8, Ex. 0.6]. Morphisms in the pointed homotopy category are given by

$$\mathrm{Hom}_{\mathsf{HoTop}_\bullet}((X, x_0), (Y, y_0)) = \{f : X \to Y \mid f \text{ continuous}\}/\simeq$$

where now $f \simeq g$ means there exists a family of continuous maps H_t for $t \in I$ defining a continuous map $H : X \times I \to Y$ such that $H_0 = f$, $H_1 = g$, and $H_t(x_0) = y_0$ for each $t \in I$. Compositions are as above. Representatives of isomorphisms in $\mathsf{HoTop}_\bullet$ are called **pointed homotopy equivalences**. Frequently, we will use the notation $\bullet \in X$ for a fixed chosen base point. Note that arrows in the homotopy categories cannot be evaluated in points. In other categories, it does not even make sense to ask about evaluating arrows. A category consists of objects and arrows only, but arrows have a direction: from domain to codomain. Therefore every category has an "opposite" obtained by flipping all arrows.

Definition 1.3

Let $\mathcal{C}$ be a category. Then the **opposite** category is the category with reversed arrows: $\mathrm{ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{ob}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$. Composition of arrows $f \circ g$ in $\mathcal{C}^{\mathrm{op}}$ is defined by $g \circ f$ in $\mathcal{C}$.

1.2 Functors

A category has objects and arrows with composition and identities. **Functors** relate one category to another. As such, they should preserve all available structure so that there is no alternative to the following definition.

Definition 1.4

A (**covariant**) **functor** $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ assigns to every $X \in \mathrm{ob}(\mathcal{C})$ an object $\mathcal{F}(X) \in \mathrm{ob}(\mathcal{D})$ and to every morphism $f : X \to Y$ with $X, Y \in \mathrm{ob}(\mathcal{C})$ a morphism $\mathcal{F}(f) \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(X), \mathcal{F}(Y))$ such that

(i) $\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f)$ for all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$.
(ii) $\mathcal{F}(\mathrm{id}_X) = \mathrm{id}_{\mathcal{F}(X)}$ for all $X \in \mathrm{ob}(\mathcal{C})$.

A **contravariant** functor f from $\mathcal{C}$ to $\mathcal{D}$ is a covariant functor from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$. Thus $\mathcal{F}$ satisfies the above properties, except that $F(f) \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(Y), \mathcal{F}(X))$ and $\mathcal{F}(g \circ f) = \mathcal{F}(f) \circ \mathcal{F}(g)$.

Example 1.5 The more abstract the definition, the more examples must be given.

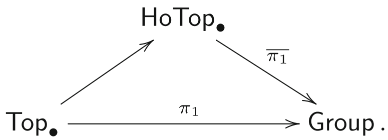
(i) Every category $\mathcal{C}$ comes with an identity functor $\mathrm{id}_{\mathcal{C}}$.

(ii) Still not very inspiring but theoretically of utmost importance are the **forgetful functors** that drop either some part of the structure or some information on the objects, like K-vect $\to$ Ab, Ab $\to$ Group, and Group $\to$ Set.

(iii) The **fundamental group** defines a functor $\pi_1 \colon \mathsf{Top}_\bullet \to \mathsf{Group}$. On objects, a pointed space (X, x_0) gives rise to the fundamental group $\pi_1(X, x_0)$. On morphisms, an arrow $(X, x_0) \to (Y, y_0)$ in $\mathsf{Top}_\bullet$ induces the group homomorphism $\pi_1(f) \colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ defined by pushing loops, $[\gamma] \mapsto [f \circ \gamma]$.

(iv) We have functors $\mathsf{Top} \to \mathsf{HoTop}$ and $\mathsf{Top}_\bullet \to \mathsf{HoTop}_\bullet$, which leave objects untouched and place morphisms in their homotopy classes.

(v) Let $f, g : (X, x_0) \to (Y, y_0)$ be homotopic pointed maps and let H be a pointed homotopy. For a loop $\gamma \colon (I, \{0, 1\}) \to (X, x_0)$ representing an element of $\pi_1(X, x_0)$, the map $(s, t) \mapsto H(\gamma(s), t)$ defines a pointed homotopy $H \colon I \times I \to Y$ from the path $f \circ \gamma$ to the path $g \circ \gamma$. Hence $\pi_1(f) = \pi_1(g)$. Thus the functor π_1 factorizes over the pointed homotopy category



Here we understood implicitly that functors $\mathcal{F} \colon \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} \colon \mathcal{D} \to \mathcal{E}$ can be **composed** to a functor $\mathcal{G} \circ \mathcal{F} \colon \mathcal{C} \to \mathcal{E}$.

(vi) **Abelianization** is a functor $(-)_{\mathrm{ab}} : \mathsf{Group} \to \mathsf{Ab}$, which takes a group G and forms the quotient group $G \mapsto G/[G, G]$ where $[G, G]$ is the **derived subgroup** generated by all **commutators** $g_1 g_2 g_1^{-1} g_2^{-1}$ for $g_1, g_2 \in G$. Since a group homomorphism sends commutators to commutators, every group homomorphism $f \colon G \to H$ induces a homomorphism of abelian groups $f_{\mathrm{ab}} \colon G_{\mathrm{ab}} \to H_{\mathrm{ab}}$. We will sometimes write the equivalence class of an element $g \in G$ as $[g]_{\mathrm{ab}} \in G_{\mathrm{ab}}$.

(vii) The **free vector space** construction $F \colon \mathsf{Set} \to K\text{-}\mathsf{vect}$ takes a set X and forms the vector space $F(X)$ of all formal K-linear combinations $\sum_{x \in X} \lambda_x x$ with $\lambda_x \in K$ different from zero for only finitely many $x \in X$. It is clear how addition and scalar multiplication are defined. Hence by construction, X is a basis of $F(X)$. A map of sets $X \to Y$ gives a K-linear map $F(X) \to F(Y)$ by unique K-linear extension.

(viii) Forming the dual vector space $V^* = \mathrm{Hom}_K(V, K)$ of a K-vector space V gives a **contravariant** functor $D \colon K\text{-}\mathsf{vect} \to K\text{-}\mathsf{vect}$. A K-linear map $f \colon V \to W$ defines a K-linear map $D(f) \colon W^* \to V^*$ by precomposing linear forms, $\varphi \mapsto \varphi \circ f$.

(ix) Fixing a K-vector space W, the tensor product $(\cdot) \otimes_K W$ is a covariant functor $K\text{-}\mathsf{vect} \to K\text{-}\mathsf{vect}$.

Recall the construction of the tensor product $V \otimes_K W$ in the last example. One starts with the free vector space $F(V \times W)$ and mods out the linear subspace spanned by all elements $(\lambda u + \mu v, w) - \lambda(u, w) - \mu(v, w)$ for $\lambda, \mu \in K$, $u, v \in V$, and $w \in W$, and similarly in the other variable, to enforce the familiar distributivity and bilinearity properties of tensors. To be pedantic, we should have applied the forgetful functor $K\text{-}\mathsf{vect} \to \mathsf{Set}$ to V and W before forming the free vector space $F(V \times W)$.

At a first encounter, the uninitiated might wonder why one works with this quotient of intimidatingly large spaces when one could instead pick bases $\{e_i\}$ and $\{f_j\}$ of V and W to define $V \otimes_K W$ as the free vector spaces spanned by $\{e_i \otimes f_j\}$. The reason is now apparent. We want $(\,\cdot\,) \otimes_K W$ to be a functor, and a functorial construction must not depend on arbitrary choices.

1.3 Natural Transformations

> "It is not too misleading, at least historically, to say that categories are what one must define in order to define functors, and that functors are what one must define in order to define natural transformations."
>
> (Peter J. Freyd)

Certainly long before the definition of category was given, mathematicians were aware that some isomorphisms were better than others. For example, if V is a finite dimensional vector space, one can pick a basis and map it to the dual basis of V^*. The unique linear extension gives an isomorphism from V to V^*. However, this isomorphism comes at the cost of a choice: We had to select a basis of V to begin with. In contrast, we get an isomorphism $V \xrightarrow{\cong} V^{**}$ for free: map a vector $v \in V$ to the linear form *on* V^* given by evaluating linear forms *from* V^* at v. One might say this isomorphism arises *naturally*, and **natural transformations** seek to capture what this naturality should mean mathematically. As an important advantage, a "naturally defined" isomorphism will automatically be compatible with homomorphisms. In our example, this means that for all linear maps $f\colon V \to W$, the diagram

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \downarrow & & \downarrow \\ V^{**} & \xrightarrow{f^{**}} & W^{**} \end{array}$$

commutes. This observation is the basis for the following definition.

Definition 1.6

A **natural transformation** $\eta : \mathcal{F} \to \mathcal{G}$ of functors $\mathcal{F}, \mathcal{G} : \mathcal{C} \to \mathcal{D}$ assigns a morphism $\eta_A : \mathcal{F}(A) \to \mathcal{G}(A)$ in $\mathcal{D}$ to each $A \in \mathrm{ob}(\mathcal{C})$ such that

$$\begin{array}{ccc} \mathcal{F}(A) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(B) \\ {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta_B} \\ \mathcal{G}(A) & \xrightarrow[\mathcal{G}(f)]{} & \mathcal{G}(B) \end{array}$$

commutes for all $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$. The morphism η_A is called the **component** of η at $A \in \mathrm{ob}(\mathcal{C})$.

Example 1.7 Let us first check that our example from the above discussion meets this definition. We consider the **double dual space functor** $DD : K\text{-vect} \to K\text{-vect}$, which maps a K-linear map $f\colon V \to W$ to the K-linear map

$$\begin{aligned} f^{**} &: V^{**} \to W^{**} \\ (\delta &: V^{*} \to K) \mapsto (\varphi \mapsto \delta(\varphi \circ f)). \end{aligned}$$

We claim we have a natural transformation $\eta : \mathrm{id}_{K\text{-vect}} \to DD$ with components

$$\eta_V \colon V \to V^{**},\ v \mapsto \begin{cases} \mathrm{ev}_v \colon V^{*} \to K \\ \varphi \mapsto \varphi(v). \end{cases}$$

So for every $f\colon V \to W$, we have to check that

$$\begin{array}{ccc} V & \xrightarrow{\;f\;} & W \\ {\scriptstyle \eta_V}\downarrow & & \downarrow{\scriptstyle \eta_W} \\ V^{**} & \xrightarrow{\;f^{**}\;} & W^{**} \end{array}$$

commutes. Indeed, let $v \in V$ and $\varphi \in W^{*}$. Then we obtain

$$f^{**} \circ \eta_V(v)(\varphi) = f^{**}(\mathrm{ev}_v)(\varphi) = \mathrm{ev}_v(\varphi \circ f) = \varphi(f(v)) = \mathrm{ev}_{f(v)}(\varphi) = \eta_W \circ f(v)(\varphi).$$

Example 1.8 Every group G can be interpreted as a category $\underline{G}$ with only one object, denote it by "$\bullet$", and morphism set $\mathrm{Hom}_{\underline{G}}(\bullet, \bullet) = G$ with composition given by multiplication in G:

$$g_1 \circ g_2 = g_1 g_2.$$

Then a functor $\mathcal{F}\colon \underline{G} \to \mathsf{Set}$ is the same as a G-set. The set is $X_{\mathcal{F}} = \mathcal{F}(\bullet)$ and the action is given by $g \cdot x := \mathcal{F}(g)(x)$ for $g \in G$ and $x \in X$. To convince ourselves that this defines an action, we have to check

(i) $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$,
(ii) $e \cdot x = x$.

To see (i), we note that $g_1 \cdot (g_2 \cdot x)$ equals

$$\mathcal{F}(g_1)(\mathcal{F}(g_2)(x)) = \mathcal{F}(g_1) \circ \mathcal{F}(g_2)(x) = \mathcal{F}(g_1 \circ g_2)(x) = \mathcal{F}(g_1 g_2)(x) = (g_1 g_2) \cdot x.$$

To see (ii), we calculate $e \cdot x = \mathcal{F}(e)(x) = \mathcal{F}(\mathrm{id}_\bullet)(x) = \mathrm{id}_{\mathcal{F}(\bullet)}(x) = \mathrm{id}_{X_{\mathcal{F}}}(x) = x$. A natural transformation $\eta : \mathcal{F} \to \mathcal{G}$ of functors $\mathcal{F}, \mathcal{G} : \underline{G} \to \mathsf{Set}$ is then the same as a G-equivariant map of G-sets. Indeed, naturality gives $\eta(g \cdot x) = g \cdot \eta(x)$ for $x \in X_{\mathcal{F}}$ and $g \in G$.

Definition 1.9

A natural transformation $\eta : \mathcal{F} \to \mathcal{G}$ of functors $\mathcal{F}, \mathcal{G} : \mathcal{C} \to \mathcal{D}$ is called a **natural isomorphism**, if all components η_A are isomorphisms for $A \in \mathcal{C}$.

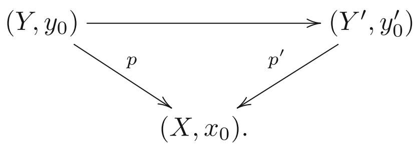
Be aware that the two functors $\mathrm{id}_{K\text{-vect}}$ and DD from K-vect to K-vect are *not* naturally isomorphic because V is not isomorphic to V^{**} if V is infinite-dimensional. But they are naturally isomorphic as functors on the category of finite dimensional K-vector spaces by the natural transformation we defined. In contrast, the functors $\mathrm{id}_{K\text{-vect}}$ and D do not even have a chance to be naturally isomorphic because $\mathrm{id}_{K\text{-vect}}$ is covariant while D is contravariant. For an example of two covariant functors that give isomorphic objects but not naturally so, see Exercise 1.2. With the notion of natural isomorphisms at hand, we can now define what it should mean that two categories are equivalent.

Definition 1.10

An **equivalence** of categories $\mathcal{C}$ and $\mathcal{D}$ consists of functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ such that $\mathcal{F} \circ \mathcal{G}$ and $\mathcal{G} \circ \mathcal{F}$ are naturally isomorphic to $\mathrm{id}_{\mathcal{D}}$ and $\mathrm{id}_{\mathcal{C}}$, respectively.

Requiring $\mathcal{F} \circ \mathcal{G} = \mathrm{id}_{\mathcal{D}}$ and $\mathcal{G} \circ \mathcal{F} = \mathrm{id}_{\mathcal{C}}$ leads to the notion of **isomorphism** of categories, which for most practical purposes is too much to ask for. We say that $\mathcal{C}$ is **dually equivalent** to $\mathcal{D}$ if $\mathcal{C}$ is equivalent to $\mathcal{D}^{\mathrm{op}}$.

Example 1.11 Let (X, x_0) be a pointed, path connected, locally path connected, and semi-locally simply connected space. We introduce the category $\mathsf{Cov}_{(X,x_0)}$ whose objects are path connected pointed covering spaces $p\colon (Y, y_0) \to (X, x_0)$ and whose morphisms are commutative triangles



We introduce a second category $\mathsf{Sub}_{\pi_1(X,x_0)}$ whose objects are subgroups of the fundamental group $\pi_1(X, x_0)$ and whose morphisms are inclusions. We have a functor $\mathrm{Char}\colon \mathsf{Cov}_{(X,x_0)} \to \mathsf{Sub}_{\pi_1(X,x_0)}$, which on objects associates the characteristic subgroup $\mathrm{Char}(p) = \mathrm{im}\,\pi_1(p)$ with a covering map p. On morphisms, a commutative triangle of covering maps induces an inclusion of characteristic subgroups as we see by applying the π_1-functor. The classification theorem of covering spaces now takes the elegant form that Char is an equivalence of categories that maps regular coverings to normal subgroups.

Constructing the inverse functor basically amounts to the existence part of the proof of the classification theorem. Given a subgroup $G \subseteq \pi_1(X, x_0)$ we construct a covering space X_G by first forming the set of all paths in X starting at x_0. Then we identify two such paths if they have the same end point, resulting in a loop whose pointed homotopy class lies in G. The end point map defines a covering map $X_G \to X$ sending the constant path to the base point x_0. This map can be used to lift the topology from X to X_G and the awkward "semi-locally simply connected" property is necessary to ensure that fibers are discrete. It is a good exercise to find the natural isomorphism from $X_{\text{Char}(\cdot)}$ to $\text{id}_{\mathsf{Cov}_{(X,x_0)}}$ *(Hint: unique path and homotopy lifting)*.

Note that no two distinct objects in $\mathsf{Sub}_{\pi_1(X,x_0)}$ are isomorphic and the functor $X_{(\,\cdot\,)}$ selects precisely one object from each isomorphism class of objects in $\mathsf{Cov}_{(X,x_0)}$. Rephrasing this, the functor $X_{(\,\cdot\,)}$ gives an algebraic description of a **skeleton** of $\mathsf{Cov}_{(X,x_0)}$. To some extent, finding algebraic skeleta of topological categories is the big hairy audacious goal of algebraic topology.

Example 1.12 A similar construction is familiar from field theory. The category of intermediate fields of a Galois extension L/K is equivalent to $\mathsf{Sub}_{\text{Gal}(L/K)}$. The equivalence sends $K \subseteq Z \subseteq L$ to the subgroup of automorphisms in $\text{Gal}(L/K)$ fixing each element of Z. Galois extensions Z/K correspond to normal subgroups of $\text{Gal}(L/K)$. The inverse functor maps $G \subseteq \text{Gal}(L/K)$ to the fixed subfield L^G.

1.4 Adjunction

Every forgetful functor asks a question. What is the most general and most efficient construction in the reverse direction? For example, take the forgetful functor K-$\mathsf{vect} \to \mathsf{Set}$. How can we efficiently turn a set into a vector space in always the same way, without using any knowledge on the particular set? Well, we apply the free vector space construction from Example (vii) in Sect. 1.2! We take a set and just artificially form linear combinations of elements of the set with coefficients from K. As another example, for the forgetful functor $\mathsf{Ab} \to \mathsf{Group}$, the general efficient reverse (*reverse, not inverse!*) construction is the abelianization from Example (vi) in Sect. 1.2.

The mathematically rigorous formulation of the vague question for a "general, efficient construction" is: Given a functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$, what is the **left adjoint** functor $\mathcal{G}\colon \mathcal{D} \to \mathcal{C}$? Of course, we can also start with a functor $\mathcal{G}$ and ask for a functor $\mathcal{F}$ so that $\mathcal{G}$ provides the most general and most efficient construction in the reverse direction. Or in mathematical terms: "Given a functor $\mathcal{G}\colon \mathcal{D} \to \mathcal{C}$, what is the **right adjoint** functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$?" Similarly as in the case of natural transformation, the key to the definition is to consider all relevant morphisms at the same time:

Definition 1.13
Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ be functors. We call $\mathcal{F}$ **left adjoint** to $\mathcal{G}$ (and $\mathcal{G}$ **right adjoint** to $\mathcal{F}$) if for all $A \in \text{ob}(\mathcal{C})$ and $B \in \text{ob}(\mathcal{D})$ we have a bijection

$$\text{Hom}_{\mathcal{D}}(\mathcal{F}(A), B) \cong \text{Hom}_{\mathcal{C}}(A, \mathcal{G}(B)), \quad f \mapsto \overline{f}$$

which is **natural** in A and B.

"Natural in A" means that for each fixed $B \in \text{ob}(\mathcal{D})$ the two functors

$$\text{Hom}_{\mathcal{D}}(\mathcal{F}(-), B),\ \text{Hom}_{\mathcal{C}}(-, \mathcal{G}(B))\colon \mathcal{C}^{\text{op}} \to \mathsf{Set}$$

are naturally isomorphic. Here and elsewhere, we assume $\mathcal{C}$ and $\mathcal{D}$ are **locally small**, meaning that morphisms from one object to another form a set and not a proper class. Decoding the definition, naturality in A means

$$\overline{\varphi} \circ f = \overline{\varphi \circ \mathcal{F}(f)} \quad \text{for } \varphi \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(A), B),\; f \in \mathrm{Hom}_{\mathcal{C}}(A', A).$$

Similarly, naturality in B gives

$$\mathcal{G}(g) \circ \overline{\varphi} = \overline{g \circ \varphi} \quad \text{for } \varphi \in \mathrm{Hom}_{D}(\mathcal{F}(A), B),\; g \in \mathrm{Hom}_{\mathcal{D}}(B, B').$$

The word *adjoint* goes back to the formal similarity of the morphism set adjunction to the defining equation of adjoint operators on Hilbert space $\langle fx, y\rangle = \langle x, f^* y\rangle$. The concept of adjunction is not reserved for forgetful functors. For these, it is however particularly enlightening to figure out the left adjoints.

Example 1.14 The free vector space functor $F\colon \mathsf{Set} \to K\text{-}\mathsf{vect}$ is left adjoint to the forgetful functor $\mathcal{G}\colon K\text{-}\mathsf{vect} \to \mathsf{Set}$. The natural isomorphism defining the adjunction

$$\mathrm{Hom}_{K\text{-}\mathsf{vect}}(F(S), V) \cong \mathrm{Hom}_{\mathsf{Set}}(S, \mathcal{G}(V)),\; f \mapsto \overline{f} = f|_S$$

is just given by restriction. Restriction is injective because linear maps are uniquely determined by the values on a basis. Restriction is surjective because any map $S \to \mathcal{G}(V)$ can be linearly extended to a K-linear map $F(S) \to V$. Naturality is clear from the above formulas.

Example 1.15 Let $\mathcal{G} : \mathsf{Group} \to \mathsf{Set}$ be the forgetful functor. It has a left-adjoint $\mathcal{F} : \mathsf{Set} \to \mathsf{Group}$, which associates with a set S the so-called **free group** on the alphabet S. As a set $\mathcal{F}(S)$ consists of all "words" such as $a^3b^{-2}aab^{-5}b^5c$ on letters $a, b, c \in S$, where we identify two such words if they can be obtained from one another by the obvious cancellations and expansions, for example $aa^{-1}b = b$, $c^2c^{-3} = c^{-1}$. Composition is given by concatenation of words. The unit element is the **empty word**, sometimes called "e" or "1." Note that no two letters $s_1, s_2 \in S$ commute in $\mathcal{F}(S)$: We have $s_1s_2 \neq s_2s_1$ unless $s_1 = s_2$.

Every map $S_1 \xrightarrow{f} S_2$ has a **unique** extension to a group homomorphism $\mathcal{F}(S_1) \xrightarrow{\mathcal{F}(f)} \mathcal{F}(S_2)$, which one obtains as in the example

$$\mathcal{F}(f)(abc^{-3}b) = f(a)f(b)f(c)^{-3}f(b).$$

Given any group H, the adjunction yields a unique morphism $\varepsilon_H : \mathcal{F}(\mathcal{G}(H)) \to H$, which restricts to the identity on the set H. In fact, ε defines a natural transformation from $\mathcal{F} \circ \mathcal{G}$ to $\mathrm{id}_{\mathsf{Group}}$, called the **counit** of the adjunction. Since ε_H is surjective,

$$\overline{\varepsilon_H} : \mathcal{F}(\mathcal{G}(H))/\ker \varepsilon_H \xrightarrow{\cong} H$$

is an isomorphism. So the upshot is that every group is isomorphic to a quotient of a free group. This is what makes free groups useful to describe general groups.

Definition 1.16
Let S be a set and let R be a subset of $\mathcal{F}(S)$. The pair (S, R) is called a **presentation** of the group G, if $G \cong \mathcal{F}(S)/\mathcal{N}(R)$, where $\mathcal{N}(R)$ is the smallest normal subgroup of $\mathcal{F}(S)$, which contains R.

The above shows that every group G has a presentation (S, R) and we use the notation $G = \langle S|R\rangle$. Elements of S are called **generators** and elements of R are called **relators**. So every relator $r \in R$ gives rise to a **relation** $r = e$, which holds true in G. Of course, taking the whole group as set of generators as above is rarely a good idea. One should rather aim for efficient presentations with S and R as small as possible. A group G is called **finitely generated**, if there exists a presentation $G = \langle S|R\rangle$ with a finite set S. It is called **finitely presented**, if there is $G = \langle S|R\rangle$ with both S and R finite.

Example 1.17 We claim that $\mathbb{Z}^2 = \langle a, b \mid [a, b]\rangle$. By the adjunction, requiring $a \mapsto (1, 0)$ and $b \mapsto (0, 1)$ defines a unique surjective homomorphism $p\colon \mathcal{F}(a, b) \to \mathbb{Z}^2$. We have $[a, b] \in \ker p$, hence $\mathcal{N}([a, b]) \subseteq \ker p$. So p descends to a homomorphism $\overline{p}\colon \mathcal{F}(a, b)/\mathcal{N}([a, b]) \to \mathbb{Z}^2$ and it remains to show that $\bar{p}$ is injective. Let $\bar{a}, \bar{b} \in \mathcal{F}(a, b)/\mathcal{N}([a, b])$ be the images of a and b and let $x = \bar{a}^{n_1}\bar{b}^{m_1} \cdots \bar{a}^{n_k}\bar{b}^{m_k} \in \ker(\bar{p})$. Then $\overline{p}(x) = 0$ gives

$$(0, 0) = n_1 \cdot (1, 0) + m_1 \cdot (0, 1) + \cdots + n_k \cdot (1, 0) + m_k \cdot (0, 1).$$

Thus $n_1 + \cdots + n_k = 0$ and $m_1 + \cdots + m_k = 0$, whence

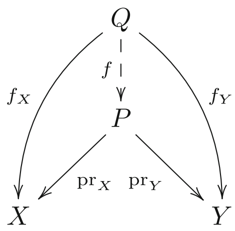
$$\bar{a}^{n_1}\bar{b}^{m_1} \cdots \bar{a}^{n_k}\bar{b}^{m_k} = \bar{a}^{n_1+\cdots+n_k}\bar{b}^{m_1+\cdots+m_k} = e.$$

1.5 Limits and Colimits

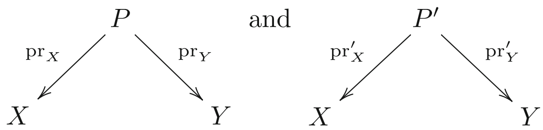
We now return to the problem we used to motivate the introduction of categories in Sect. 1.1: the abstract construction of products. The key observation is that every product (of sets, groups, topological spaces, ...) comes with **projections**

$$\begin{array}{ccc} & X \times Y & \\ {}^{\mathrm{pr}_X}\swarrow & & \searrow^{\mathrm{pr}_Y} \\ X & & Y. \end{array}$$

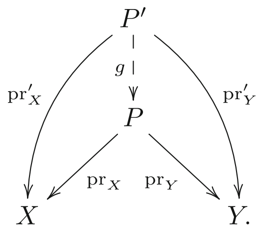
Definition 1.18
Let $\mathcal{C}$ be a category and let $X, Y \in \mathrm{ob}(\mathcal{C})$. A **product** of X and Y consists of an object $P \in \mathrm{ob}(\mathcal{C})$ and morphisms $\mathrm{pr}_X\colon P \to X$ and $\mathrm{pr}_Y\colon P \to Y$, which are required to be **universal** in the following sense. For any other object $Q \in \mathrm{ob}(\mathcal{C})$ with morphisms $f_X\colon Q \to X$, $f_Y\colon Q \to Y$, there exists a **unique** morphism $f\colon Q \to P$ such that $f_X = \mathrm{pr}_X \circ f$ and $f_Y = \mathrm{pr}_Y \circ f$.



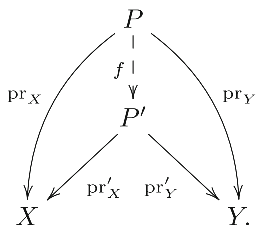
Since the unique morphism f is completely determined by f_X and f_Y, it is also customary to write $f = f_X \times f_Y$ or $f = (f_X, f_Y)$. Products might not exist in $\mathcal{C}$ (think of the category of fields), but if they do, they are **unique up to a unique isomorphism**, which transforms the projections into one another. This can be seen as follows. Say



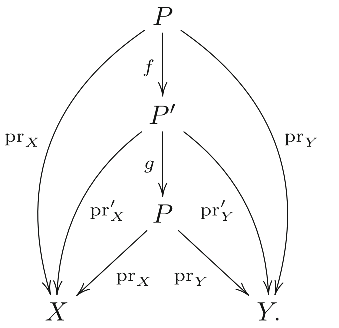
are products of X and Y. Since P is a product, we have a unique morphism $g\colon P' \to P$ such that



Similarly, since P' is a product, there exists a unique morphism $f\colon P \to P'$ with



We can compose the two diagrams as follows:



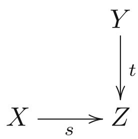
For the outer part of the diagram, uniqueness in the universal property of P says that we must have $g \circ f = \mathrm{id}_P$. Composing the diagrams in reverse order, we get $f \circ g = \mathrm{id}_{P'}$. Norman Steenrod has coined the term **abstract nonsense** for this type of purely categorical arguments. So the universal property determines a product uniquely in the strongest possible sense. It is however conceptually important to keep in mind that the product of objects X and Y is a concrete object P with projection maps p_X and p_Y (not an isomorphism class of objects).

Example 1.19 In Set, K-vect, Group, R-mod, Top, and HoTop, the categorical product of objects X and Y is the ordinary product $X \times Y$ with the usual projections. A little caution is necessary in $\mathsf{Top}^{(2)}$. The categorical product of (X, A) and (Y, B) is $(X \times Y, A \times B)$ but some authors use the notation $(X, A) \times (Y, B)$ to refer to $(X \times Y, (X \times B) \cup (A \times Y))$ instead.

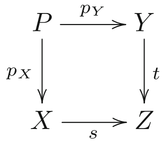
The notation $f \times g$ is often also used for arrows $f : A \to X$ and $g : B \to Y$ with different domain in a category $\mathcal{C}$ with products. In that case, it should be read as shorthand notation for what would categorically be $(f \circ \mathrm{pr}_A) \times (g \circ \mathrm{pr}_B)$. In the categories from Example 1.19, this just means that $f \times g : A \times B \to X \times Y$ is defined by $(f \times g)(a, b) = (f(a), g(b))$. To avoid any potential for confusion, it is best practice to always state the domain of $f \times g$ explicitly.

It turns out that products are only one special case of a more general construction, which goes by the name of **pullbacks**.

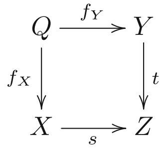
Definition 1.20
A **pullback** of a diagram



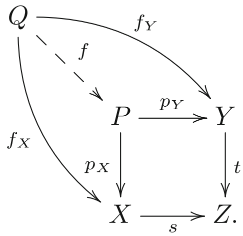
in a category $\mathcal{C}$ consists of an object $P \in \mathrm{ob}(\mathcal{C})$ and morphisms $p_X : P \to X$ and $p_Y : P \to Y$ such that the square



commutes and satisfies the following **universal property**. For every other $Q \in \mathrm{ob}(\mathcal{C})$ with morphisms $f_X \colon Q \to X$ and $f_Y \colon Q \to Y$ such that

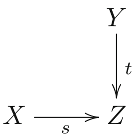


commutes, there exists a **unique** morphism $f \colon Q \to P$ such that $f_X = p_X \circ f$ and $f_Y = p_Y \circ f$.



The same abstract nonsense as above shows that pullbacks, if they exist, are unique up to a unique isomorphism that mediates between the pullback squares.

Example 1.21 In Set the pullback of any diagram



always exists. It can be constructed as

$$P = \{(x, y) \in X \times Y \colon s(x) = t(y)\}$$

together with the morphisms p_X and p_Y given by restricting the projections of the product $X \times Y$. Given a second diagram $X \overset{f_X}{\leftarrow} Q \overset{f_Y}{\rightarrow} Y$ as above, the required morphism $f \colon Q \to P$ is given by

$$f(q) = (f_X(q), f_Y(q))$$

for $q \in Q$.

Example 1.22 In Top, the pullback is the same as above set-theoretically and P carries the subspace topology of the product topology.

Let B be a topological space. Dropping base points from the category introduced in Example 1.11, we obtain the category Cov_B of covering spaces over B. So objects are covering spaces $E \to B$ and morphisms are commutative triangles

$$\begin{array}{ccc} E & \longrightarrow & E' \\ & \searrow \quad \swarrow & \\ & B. & \end{array}$$

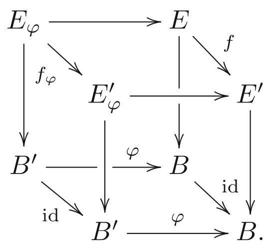
Pullback along a continuous map $\varphi\colon B' \to B$ defines a functor $\mathsf{Cov}_B \to \mathsf{Cov}_{B'}$. On objects, the functor assigns $E_\varphi \to B'$ to $E \to B$ where $E_\varphi \to B'$ sits in the pullback square

$$\begin{array}{ccc} E_\varphi & \longrightarrow & E \\ \downarrow & & \downarrow \\ B' & \xrightarrow{\varphi} & B \end{array} \tag{1.23}$$

which is obtained as in Example 1.22. On morphisms we obtain

$$\begin{array}{ccc} E_\varphi & \xrightarrow{f_\varphi} & E'_\varphi \\ & \searrow \quad \swarrow & \\ & B & \end{array} \qquad \text{from} \qquad \begin{array}{ccc} E & \longrightarrow & E' \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

where the arrow f_φ is defined by the universal property of the pullback square that occurs as the front face of the commutative cube



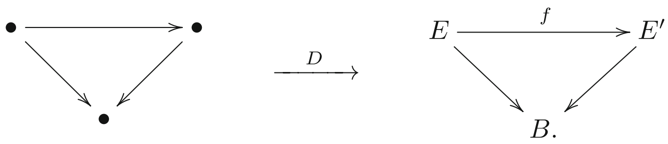
We have to check that $E_\varphi \to B'$ is indeed a covering space. If φ happens to be the inclusion of a subspace, $E_\varphi = p^{-1}(B')$ is the restricted covering. You will handle the general case in Exercise 1.4 where a hint is given.

The notion of pullback allows for yet another generalization that we introduce over three definitions. First, we still owe a formal definition of a **diagram**, a notion that we already employed several times. A category is called **small** if both all objects and all morphisms form a set and not a proper class.

Definition 1.24
Let $\mathcal{C}$ be a category and let I be a small category. A functor $D\colon I \to \mathcal{C}$ is called a **diagram** of shape I.

Other authors would call this a "small diagram," but we will have no reason to consider any other diagrams so that we require smallness right away. The category I in the diagram is also called **index category**. It is irrelevant how objects and morphisms are concretely realized in I, all that matters is the directed graph of all morphisms it defines, which thus forms the "shape" of the diagram.

Example 1.25 The above triangle of covering maps would be a diagram in Top

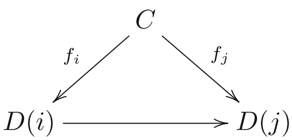


Here the left hand diagram is a complete description of the category I. The category consists of three objects, hence also three identity morphisms that are left out in the picture, and three more arrows where two of them compose to the third.

Definition 1.26
A **cone** on the diagram $D\colon I \to \mathcal{C}$ consists of an object $C \in \mathrm{ob}(\mathcal{C})$ and morphisms

$$(C \xrightarrow{f_i} D(i))_{i \in \mathrm{ob}(I)}$$

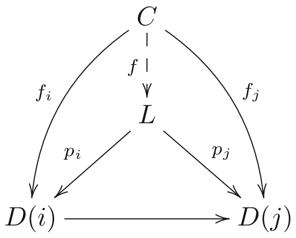
such that the triangle



commutes for all morphisms $i \to j$ in I.

The terminology is self-explanatory. One should think of the diagram as lying out flat on a table top, whereas the cone object C hovers in some distance above the table top and comes with arrows to all objects in the diagram so that the whole formation looks like a cone.

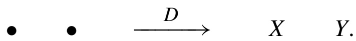
Definition 1.27
A **limit** $(L \xrightarrow{p_i} D(i))_{i \in \text{ob}\, I}$ of a diagram $D\colon I \to \mathcal{C}$ is a cone on D which is **universal** in the following sense. For every other cone $(C \xrightarrow{f_i} D(i))_{i \in \text{ob}\, I}$ there exists a **unique** morphism $f\colon C \to L$ such that $f_i = p_i \circ f$ for all $i \in \text{ob}(I)$. Thus the diagram



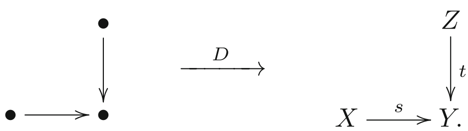
commutes for all morphisms $i \to j$ in I.

As a concept defined by a universal property, also limits are unique up to a unique isomorphism mediating between the cones. We introduce the notation $L = \lim D$.

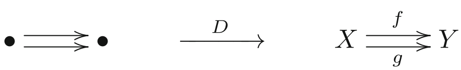
Example 1.28 The product of $X, Y \in \text{ob}(\mathcal{C})$ is the limit of the diagram



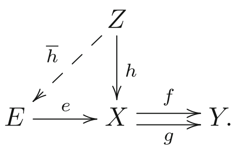
More generally, possibly infinite products can be formed by taking any small **discrete category** (with only identity morphisms) as index category I. The pullback of $X \xrightarrow{s} Y \xleftarrow{t} Z$ in $\mathcal{C}$ is the limit of the diagram



Example 1.29 The limit of the diagram



is called the **equalizer** $e\colon E \to X$ of f and g. The universal property is captured by the diagram



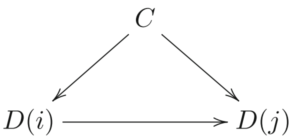
In words, for every morphism $h\colon Z \to X$, there exists a unique arrow $\overline{h}\colon Z \to E$ such that $h = e \circ \overline{h}$. In Set, the equalizer of f and g is just the inclusion $e\colon E \subseteq X$ of the subset $E = \{x \in X\colon f(x) = g(x)\}$. Indeed, any cone $h\colon Z \to X$ satisfies $f \circ h = g \circ h$, so h maps to E, hence factorizes uniquely through the inclusion e. In Top, the same argument shows that the equalizer is the inclusion $e\colon E \subseteq X$ as subspace, meaning E carries the subspace topology of X. If moreover Y is Hausdorff, then the equalizer $e\colon E \subseteq X$ is a closed embedding. To see that, just note that the Hausdorff property of Y says that the diagonal subspace $D = \{(y, y)\colon y \in Y\} \subseteq Y \times Y$ is closed in the product topology, hence so is $k^{-1}(D) = E \subseteq X$ where $k\colon X \to Y \times Y$ is the continuous product map defined by $k(x) = (f(x), g(x))$. Of course the equalizer concept generalizes in a straightforward manner to more than two arrows.

Unraveling the concepts, we see that the pullback in Example 1.28 is nothing but the equalizer of $s \circ \mathrm{pr}_X\colon X \times Z \to Y$ and $t \circ \mathrm{pr}_Z\colon X \times Z \to Y$. It turns out that this gives the correct intuition for general limits, too. If a category $\mathcal{C}$ has all binary equalizers and arbitrary products, then the limit of a diagram $D\colon I \to \mathcal{C}$ can be constructed as the equalizer of the two arrows

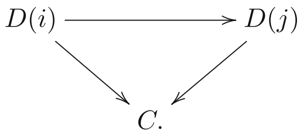
$$\prod_{i\in \mathrm{ob}(I)} D(i) \underset{t}{\overset{s}{\rightrightarrows}} \prod_{f\colon j\to k} D(k)$$

where the second product ranges over all morphisms in I and at the morphism $f\colon j \to k$, the arrow s has component $D(f) \circ \mathrm{pr}_{D(j)}$ while t has component $\mathrm{pr}_{D(k)}$. This construction shows that a category has all limits if and only if it has all products and binary equalizers.

Categorical notions have **dual** notions obtained by reversing arrows. Dual to the notion of a cone on $D\colon I \to C$ is the notion of a **cocone** from D where the diagram

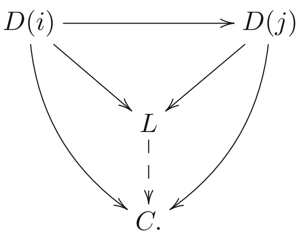


dualizes to



Definition 1.30

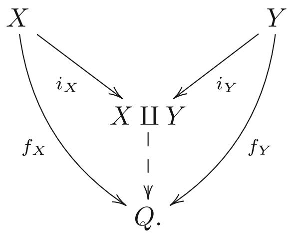
A **colimit** $(D(i) \to L)_{i\in I}$ is a cocone that is universal in the sense that every other cocone $(D(i) \to C)_{i\in \mathrm{ob}\, I}$ of D factorizes uniquely through L as indicated in the diagram



Colimits are unique up to a unique isomorphism mediating between cocones. We write $L = \operatorname{colim} D$. The colimit of the diagram

$$\bullet \quad \bullet \quad \xrightarrow{D} \quad X \quad Y$$

is called the **coproduct** $X \amalg Y$ of X and Y. Thus $X \amalg Y$ comes with morphisms $X \stackrel{i_X}{\to} X \amalg Y$ and $Y \stackrel{i_Y}{\to} X \amalg Y$ satisfying the universal property



Again, products with several, including infinitely many factors can be formed by means of larger discrete index categories.

Example 1.31 In Set, the coproduct of X and Y always exists and is given by the disjoint union of X and Y. In Top and HoTop, the coproduct of X and Y is the familiar **topological sum** of X and Y. In $\mathsf{Top}_\bullet$ and $\mathsf{HoTop}_\bullet$, the coproduct of $(X, \bullet)$ and $(Y, \bullet)$ is the **one point union** or **wedge sum** $(X \vee Y, \bullet)$.

Example 1.32 In R-mod, the coproduct of M and N is the **direct sum** $M \oplus N$ with the inclusions given by $i_M(m) = m \oplus 0$ and $i_N(n) = 0 \oplus n$. As objects we have $M \oplus N \cong M \times N$ but be aware that for an infinite index category I, coproducts $\bigoplus_{i \in \operatorname{ob} I} M_i$ and products $\prod_{i \in \operatorname{ob} I} M_i$ are not isomorphic in general. Coproducts consist only of *finite* formal sums of elements $x_i \in M_i$.

We will write $f_X \coprod f_Y$ for the unique arrow $X \coprod Y \to Q$ determined by f_X and f_Y and correspondingly "$f_X \vee f_Y$" or "$f_X \oplus f_Y$" when working in concrete categories like $\mathsf{Top}_\bullet$ or K-vect. Dually to the case of products, these notations are sometimes also used for arrows with different codomains $f : X \to A$ and $g : Y \to B$ in which case $f \coprod g$ denotes what would categorically be $(i_A \circ f) \coprod (i_B \circ g)$.

Example 1.33 For morphisms $f_M : M \to Q$ and $f_N : N \to Q$ in R-mod, we have $(f_M \oplus f_N)(m \oplus n) = f_M(m) + f_N(n)$. Indeed, taking this formula as the definition of the morphism $f_M \oplus f_N : M \oplus N \to Q$, we have

$$(f_M \oplus f_N)(i_M(m)) = (f_M \oplus f_N)(m \oplus 0) = f_M(m) + f_N(0) = f_M(m)$$

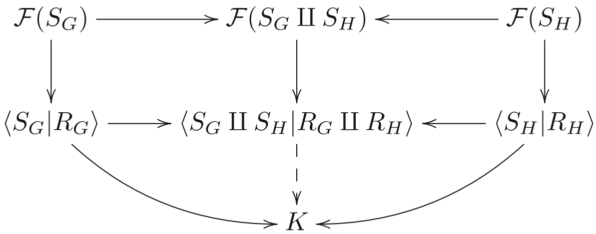
and similarly for i_N, so the required commutativity relations hold true. This proves the existence part of the universal property. If $h\colon M \oplus N \to Q$ is another morphism satisfying the commutativity relations of the coproduct, we have

$$h(m \oplus n) = h(m \oplus 0 + 0 \oplus n) = h(m \oplus 0) + h(0 \oplus n) = h(i_M(m)) + h(i_N(n)) =$$
$$= f_M(m) + f_N(n) = (f_M \oplus f_N)(m \oplus n)$$

which proves the uniqueness part of the universal property. These considerations also justify the notation $f_M + f_N$ instead of $f_M \oplus f_N$ and accordingly $f_M - f_N$ for what would categorically be $f_M \oplus (-f_N)$.

Lemma 1.34
Coproducts exist in Group. *The coproduct of G and H is called the* ***free product*** $G * H$ *of G and H. If $G \cong \langle S_G | R_G \rangle$ and $H \cong \langle S_H | R_H \rangle$, then $G * H \cong \langle S_G \amalg S_H | R_G \amalg R_H \rangle$.*

Proof We have to verify the universal property. To this end, consider the commutative diagram



The two curved arrows ending in K are given. From these we need to construct the lower vertical arrow ending in K and show that it is unique. To do so, we precompose the curved arrows with the canonical projections $\mathcal{F}(S_G) \to \langle S_G | R_G \rangle$ and $\mathcal{F}(S_H) \to \langle S_H | R_H \rangle$ to obtain the two morphisms $f\colon \mathcal{F}(S_G) \to K$ and $g\colon \mathcal{F}(S_H) \to K$. Following the forgetful-free adjunction Group $\longleftrightarrow$ Set, the arrows f and g in Group restrict to arrows $S_G \to K$ and $S_H \to K$ in Set. These two factorize over a unique map $S_G \amalg S_H \to K$ by the universal property of the coproduct in Set. Again by the forgetful-free adjunction, this map extends uniquely to a group homomorphism $u\colon \mathcal{F}(S_G \amalg S_H) \to K$ through which f and g factorize. Since f and g also factorize over the groups $\langle S_G | R_G \rangle$ and $\langle S_H | R_H \rangle$, respectively, the morphism u factorizes over a morphism $\overline{u}\colon \langle S_G \amalg S_H | R_G \amalg R_H \rangle \to K$. Since u and hence $\overline{u}$ are determined on the generating sets S_G and S_H by f and g, the morphism $\overline{u}$ is uniquely determined. □

The colimit of a diagram

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ \downarrow & & \\ \bullet & & \end{array} \quad \xrightarrow{\;D\;} \quad \begin{array}{ccc} A & \xrightarrow{t} & Y \\ {\scriptstyle s}\downarrow & & \\ X & & \end{array}$$

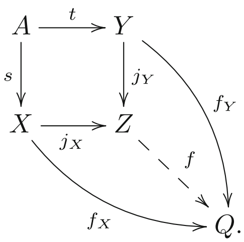
is called a **pushout** in $\mathcal{C}$. Thus the pushout consists of an object $Z \in \mathrm{ob}(\mathcal{C})$ with morphisms $X \xrightarrow{j_X} Z$, $Y \xrightarrow{j_Y} Z$ such that

$$\begin{array}{ccc} A & \xrightarrow{t} & Y \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle j_Y} \\ X & \xrightarrow[j_X]{} & Z \end{array}$$

commutes and such that for any other $Q \in \mathrm{ob}(\mathcal{C})$ that sits in a commutative square

$$\begin{array}{ccc} A & \xrightarrow{t} & Y \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle f_Y} \\ X & \xrightarrow[f_X]{} & Q, \end{array}$$

there exists a unique morphism $f : Z \to Q$ with $f_X = f \circ j_X$ and $f_Y = f \circ j_Y$.

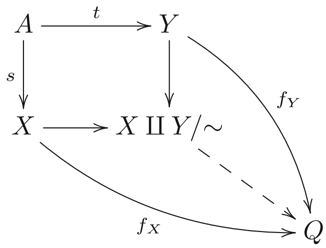


Lemma 1.35
Pushouts exist in Top *and are given by*

$$\begin{array}{ccc} A & \xrightarrow{t} & Y \\ {\scriptstyle s}\downarrow & & \downarrow \\ X & \longrightarrow & X \amalg Y/\!\sim \end{array}$$

where "$\sim$" is the finest equivalence relation on $X \amalg Y$ such that $s(a) \sim t(a)$ for all $a \in A$ and $X \amalg Y/\!\sim$ is endowed with the quotient topology with respect to "$\sim$".

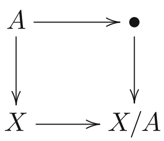
Proof We have to construct the dashed arrow in



and show it is unique. The coproduct map $f_X \amalg f_Y\colon X \amalg Y \longrightarrow Q$ sends equivalent points in $X \amalg Y$ to the same point in Q, hence it descends to a continuous map $f\colon X \amalg Y/{\sim} \longrightarrow Q$ by the universal property of the quotient topology in Appendix A, so f provides such a dashed arrow. On the other hand, any dashed arrow as above factorizes $f_X \amalg f_Y$ through $X \amalg Y/{\sim}$, so it is equal to f by the uniqueness statement of the universal property. □

In typical situations, the map s will be the inclusion of a subspace in which case we call $X \coprod Y/\sim$ an **attaching space** or **adjunction space**. Some particular examples of adjunction spaces in Top are of fundamental importance in algebraic topology, so let us study them in some detail.

Example 1.36 (Collapsing a Subspace) For a subspace $A \subseteq X$ we define the **collapsed space** X/A by the pushout



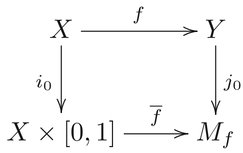
(1.37)

The subspace A becomes a single point in the collapsed space X/A, which can serve as a base point. Hence collapsing the subspace defines a functor $\mathsf{Top}^{(2)} \to \mathsf{Top}_\bullet$. Note that for $A = \emptyset$, we obtain $X/A = X \coprod \bullet$. So $X/\emptyset$ is the space obtained from X by adding a disjoint base point. In particular, $\emptyset/\emptyset = \bullet$. Another important example is the **cone** CX of a space X defined by

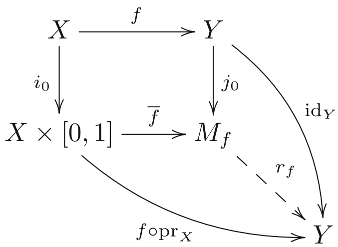
$$CX = X \times [0, 1] \,/\, X \times \{1\}.$$

The new base point in CX is the "cone tip." It is a strong deformation retract of CX. So by the **base inclusion** $X \to CX$ sending $x \in X$ to $[(x, 0)] \in CX$, every space embeds in a contractible one. We have $D^n/S^{n-1} \cong S^n$ and $CS^n \cong D^{n+1}$.

Example 1.38 (Mapping Cylinders) For a morphism $f: X \to Y$ in Top, we define the **mapping cylinder** M_f of f by the pushout



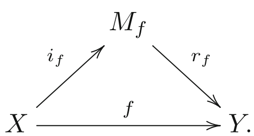
where $i_0(x) = (x, 0)$. The map $i_f: X \to M_f$ defined by $i_f(x) = \overline{f}(x, 1)$ includes X as a closed subspace of M_f. This is intuitively clear but let us not get sloppy early on and give a formal argument instead. Since i_f is a continuous injection, we only need to show every closed $C \subseteq X$ has closed image $i_f(C) \subseteq M_f$. By construction, the map $\overline{f} \amalg j_0: X \times [0, 1] \amalg Y \to M_f$ is a quotient map. Since $C \subseteq X$ is closed, so is the subset $C \times \{1\} = (\overline{f} \amalg j_0)^{-1}(i_f(C))$ of $X \times [0, 1] \amalg Y$. Hence $i_f(C) \subseteq M_f$ is closed by Lemma A.1. The universal property of the defining pushout yields a retraction



such that $r_f \circ j_0 = \mathrm{id}_Y$ and $j_0 \circ r_f \simeq_H \mathrm{id}_{M_f}$ with $H: M_f \times I \to M_f$ defined on $\mathrm{im}\,\overline{f}$ by $H(([x, t], s)) = [x, s \cdot t]$ and on $\mathrm{im}\, j_0$ by $H([y], s) = [y]$. Hence r_f is a homotopy equivalence (in fact, H is a strong deformation retraction) and for $x \in X$, we have

$$r_f(i_f(x)) = r_f(\overline{f}(x, 1)) = f(\mathrm{pr}_X(x, 1)) = f(x).$$

Thus we have proven that the category Top has the peculiar property that every arrow is the composition of a closed inclusion and a homotopy equivalence,



Example 1.39 (Mapping Cones) Combining the last two constructions, we obtain the **mapping cone** of $f: X \to Y$ defined by

$$C_f = M_f / \mathrm{im}\, i_f.$$

Equivalently, we can execute the two identifications in the other order and define C_f by the pushout

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ CX & \longrightarrow & C_f \end{array}$$

where the left vertical arrow is the base inclusion. Again, the cone tip, given by the collapsed space $\operatorname{im} i_f$ in the first definition, can serve as a base point so that C_f becomes an object of $\mathsf{Top}_\bullet$. One may think of the mapping cone C_f as a homotopy theoretically better behaved version of the collapse space $Y/\operatorname{im} f$.

Example 1.40 (Attaching an n-Cell Along f) An **attaching map** $f\colon S^{n-1} \to Y$ defines the pushout

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ D^n & \longrightarrow & Z. \end{array}$$

We say Z is obtained from Y by **attaching an n-cell** along f. For example

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & D^n \\ \downarrow & & \downarrow \\ D^n & \longrightarrow & S^n. \end{array}$$

Pushouts in Top have the following permanence property with respect to products.

Proposition 1.41
If K is a locally compact space and

$$\begin{array}{ccc} A & \xrightarrow{f_2} & Y \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle g_2} \\ X & \xrightarrow{g_1} & Z \end{array}$$

is a pushout square in Top, then so is

$$\begin{array}{ccc} A \times K & \xrightarrow{f_2 \times \mathrm{id}_K} & Y \times K \\ {\scriptstyle f_1 \times \mathrm{id}_K}\downarrow & & \downarrow{\scriptstyle g_2 \times \mathrm{id}_K} \\ X \times K & \xrightarrow{g_1 \times \mathrm{id}_K} & Z \times K. \end{array}$$

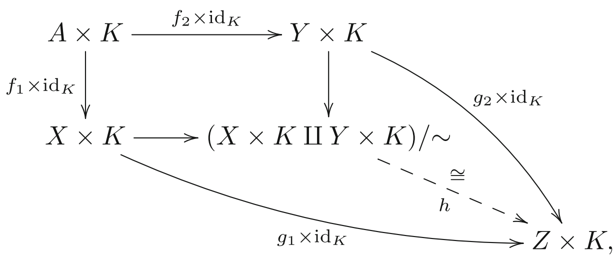
Proof By Lemma 1.35, the coproduct map $g_1 \amalg g_1 \colon X \amalg Y \longrightarrow Z$ is an identification map. Hence by Proposition A.2(ii), the map

$$g_1 \times \mathrm{id}_K \amalg g_2 \times \mathrm{id}_K \colon\ X \times K \ \amalg\ Y \times K \xrightarrow{\qquad\qquad} Z \times K$$

is an identification map, too. Therefore Lemma A.1 shows that the map

$$h = \overline{g_1 \times \mathrm{id}_K \amalg g_2 \times \mathrm{id}_K} \colon\ (X \times K \ \amalg\ Y \times K)/{\sim} \xrightarrow{\qquad\qquad} Z \times K$$

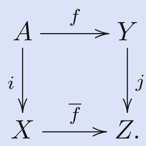
is a homeomorphism where $(x, k_1) \sim (y, k_2)$ if and only if $g_1(x) = g_2(y)$ and $k_1 = k_2$. This in turn is equivalent to the existence of some $a \in A$ with $f_1(a) = x$, $f_2(a) = y$ and again $k_1 = k_2$. Thus the homeomorphism h is precisely the unique arrow in the pushout diagram



showing that the outer square is a pushout itself. □

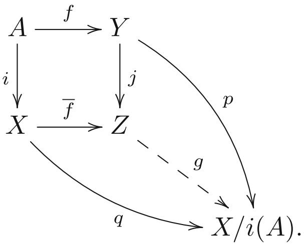
Corroborating the intuition that pushouts in Top describe gluings of spaces, they exhibit the following three convenient properties.

Theorem 1.42
Consider a pushout square in Top



(i) The map $\overline{f}$ descends to a homeomorphism $X/i(A) \cong Z/j(Y)$.
(ii) If i is the inclusion of a closed subspace, then so is j. In that case $\overline{f}$ restricts to a homeomorphism $X \setminus i(A) \cong Z \setminus j(Y)$.
(iii) If i is the inclusion of a strong deformation retract, then so is j.

Proof *Part (i).* By commutativity of the square, the map $\overline{f}$ descends to a map of sets $\widetilde{f}\colon X/i(A) \to Z/j(Y)$. The universal property of the quotient space $X/i(A)$ implies that $\widetilde{f}$ is continuous. To construct the inverse of $\widetilde{f}$, consider the diagram

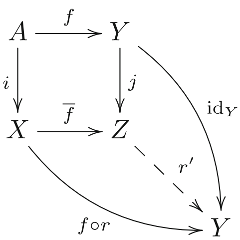


in which q is the quotient map and p is the constant map with value the base point $i(A) \in X/i(A)$ (which also exists if $A = \emptyset$). Clearly, q and p form a cocone on i and f so we get a unique map $g\colon Z \to X/i(A)$ such that the entire diagram commutes. Since $p = g \circ j$, the map g descends to a map of sets $\overline{g}\colon Z/j(Y) \to X/i(A)$, which is continuous by the universal property of the quotient space $Z/j(Y)$. The identities $\overline{g} \circ \widetilde{f} = \mathrm{id}_{X/i(A)}$ and $\widetilde{f} \circ \overline{g} = \mathrm{id}_{Z/j(Y)}$ hold by construction.

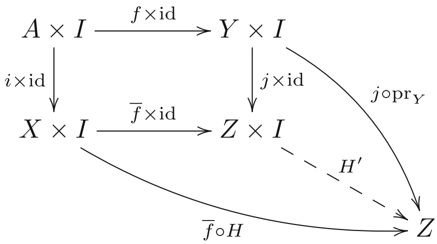
Part (ii). First we observe that j is injective because two different points in Y only get identified in Z if both lie in the image of f and if their preimages are mapped to the same point in X via i. But this does not happen because i is injective. To see that j is the inclusion of a closed subset, it remains to show that j is a closed map. So let $C \subseteq Y$ be closed. Then $(\overline{f} \amalg j)^{-1}(j(C)) = i(f^{-1}(C)) \amalg C$ is closed in $X \amalg Y$ because f is continuous and i is a closed map. Since $\overline{f} \amalg j$ is an identification map, it follows from Lemma A.1 that $j(C)$ is closed.

Knowing that both i and j are inclusions of closed subspaces, we can now treat them as such and suppress the letters i and j in the notation. To prove the second statement of (ii), we infer from (i) that $X/A \setminus A/A \cong Z/Y \setminus Y/Y$. From this the assertion follows once we verify that the bijections $X \setminus A \longrightarrow X/A \setminus A/A$ and $Z \setminus Y \longrightarrow Z/Y \setminus Y/Y$, arising as restrictions of $p\colon X \longrightarrow X/A$ and $q\colon Z \longrightarrow Z/Y$, are open maps. To do so, note that an open set $U \subseteq X \setminus A$ is also open in X because A is closed. Since p is an identification map and $U = p^{-1}(p(U))$, it follows from Lemma A.1 that $p(U)$ is open in X/A, hence also in $X/A \setminus A/A$. The same argument applies to $Z \setminus Y \longrightarrow Z/Y \setminus Y/Y$.

Part (iii). Let $r\colon X \to A$ be a retraction and let $H\colon X \times I \to X$ be a homotopy with $H_0 = \mathrm{id}_X$, $H_1 = i \circ r$, and $H_t(i(a)) = i(a)$ for $a \in A$. The pushout



gives a retraction $r'\colon Z \to Y$ of j. Using Proposition 1.41, the pushout



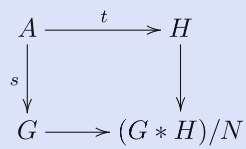
yields a homotopy H'. Indeed, the bent arrows form a cocone because the deformation retraction H is strong. We check that H' is a deformation retraction $\mathrm{id}_Z \simeq_{H'} j \circ r'$. Indeed, each $z \in Z$ either lies in $\operatorname{im} \overline{f}$ or in $\operatorname{im} j$ (or in both) and

$$H_0'(j(y)) = j(\mathrm{pr}_Y((y,0))) = j(y), \quad H_0'(\overline{f}(x)) = \overline{f}(H_0(x)) = \overline{f}(x),$$
$$H_1'(j(y)) = j(\mathrm{pr}_Y((y,1))) = j(y) = j(r'(j(y))),$$
$$H_1'(\overline{f}(x)) = \overline{f}(H_1(x)) = \overline{f}(i(r(x))) = j(f(r(x))) = j(r'(\overline{f}(x))).$$

Finally, H' is strong: $H'(j(y),t) = H'((j \times \mathrm{id})(y,t)) = j(\mathrm{pr}_Y(y,t)) = j(y)$. □

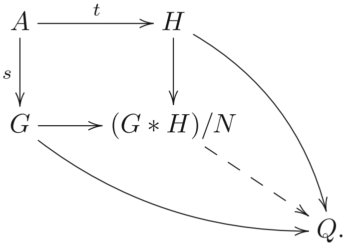
We remark that pushouts in HoTop do not always exist. Instead, one works with a non-categorical substitute called "homotopy pushouts" (and more generally "homotopy colimits") as we discuss later in Sect. 2.3. In the category Group, however, pushouts exist and are constructed as quotients of coproducts much like in the category Top.

Lemma 1.43
Pushouts exist in Group *and are given by*



where $N = \mathcal{N}(\{s(a)t(a)^{-1}\colon a \in A\})$ *is the smallest normal subgroup of* $G * H$ *containing* $s(a) \cdot t(a)^{-1}$ *for all* $a \in A$.

Proof We have to show existence and uniqueness of the dashed arrow in



The arrow $G * H \to Q$ from the universal property of the coproduct descends to $(G * H)/N$ by commutativity of the diagram. Uniqueness follows from the universal property of the quotient group that is formally the same as the universal property of the quotient topology used in Lemma 1.35. □

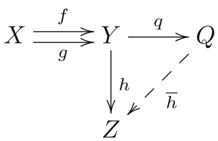
In the special case that s and t are injective, the pushout group $(G * H)/N$ is called the free product of G and H with **amalgamation** over A and is denoted by $G *_A H$. If $G = \langle S_1 | R_1 \rangle$ and $H = \langle S_2 | R_2 \rangle$, then a presentation of the pushout group $(G * H)/N$ is given by

$$\langle S_1 \amalg S_2 \mid R_1 \amalg R_2 \amalg \{s(a)t(a)^{-1} \colon a \in A\} \rangle.$$

To complete the picture, let us now consider the diagram

$$\bullet \rightrightarrows \bullet \qquad \xrightarrow{\;D\;} \qquad X \overset{f}{\underset{g}{\rightrightarrows}} Y$$

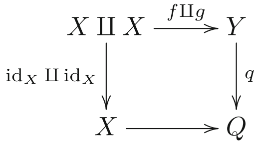
whose colimit is called the **coequalizer** $q \colon Y \to Q$ of f and g. The universal property is captured by the diagram



In words, for every morphism $h \colon Y \to Z$, there exists a unique arrow $\overline{h} \colon Q \to Z$ such that $h = \overline{h} \circ q$. In other words,

$$X \overset{f}{\underset{g}{\rightrightarrows}} Y \xrightarrow{\;q\;} Q$$

is a coequalizer diagram if and only if



is a pushout diagram. From this we see that in **Set**, the coequalizer of f and g is the quotient map $q: Y \to Q$ where $Q = Y/{\sim}$ is the set of equivalence classes for the finest equivalence relation on Q, which asserts that $f(x) \sim g(x)$ holds true for all $x \in X$. In Top, the coequalizer has the same description as in Set where now Q carries the quotient topology with respect to "$\sim$".

The pushout of $X \xleftarrow{s} A \xrightarrow{t} Y$ in a category $\mathcal{C}$ is the coequalizer of the arrows $i_X \circ s: A \to X \coprod Y$ and $i_Y \circ t: A \to X \coprod Y$ and the general description of colimits should by now be apparent. If $\mathcal{C}$ has all binary coequalizers and all coproducts, then the colimit of the diagram $D: I \to \mathcal{C}$ is just the coequalizer of

$$\coprod_{f:\, i \to j} D(i) \overset{s}{\underset{t}{\rightrightarrows}} \coprod_{k \in \mathrm{ob}(I)} D(k)$$

where the cocomponent of s at the morphism $f: i \to j$ is the inclusion $i_{D(i)}$ while the cocomponent of t is $i_{D(j)} \circ D(f)$. Hence a category has all colimits if and only if it has all coproducts and binary coequalizers.

In the category Top, we thus see that not only a pushout but also a general colimit of a diagram $D: I \to \mathsf{Top}$ is given by a quotient space of the topological sum $\coprod_{i \in \mathrm{ob}(I)} D(i)$. In particular, the proof of Proposition 1.41 carries over to general colimits so that we have the following result.

Proposition 1.44
Let K be a locally compact space and consider the functor $\Pi_K: \mathsf{Top} \to \mathsf{Top}$, which sends a map $f: X \to Y$ to $f \times \mathrm{id}_K: X \times K \to Y \times K$. Then Π_K is ***cocontinuous****: for every diagram $D: I \to \mathsf{Top}$, we have*

$$\mathrm{colim}_{i \in I}\, \Pi_K(D(i)) = \Pi_K(\mathrm{colim}_{i \in I}\, D(i)).$$

As an alternative proof, one can show that the mapping space functor $(-)^K$, which takes a space X to the space of continuous maps $X^K = \{f: K \to X\}$ with the compact open topology, is right adjoint to Π_K and apply Exercise 1.5 below.

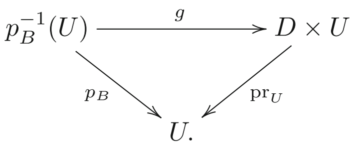
Exercises

1.1 Let G be a group and denote by G-Set the category of left G-sets. Recall that objects are functors from $\underline{G}$ to Set and morphisms are natural transformations. Find the left adjoint to the forgetful functor $G\text{-}\mathsf{Set} \to \mathsf{Set}$.

1.2 Let Fin-Bij be the category whose objects are finite sets and whose morphisms are bijections. For a finite set X, let $\mathcal{B}(X)$ be the set of bijections $X \to X$ and let $\mathcal{O}(X)$ be the set of total orders on X. Turn $\mathcal{B}$ and $\mathcal{O}$ into functors $\mathsf{Fin\text{-}Bij} \to \mathsf{Set}$. Show that there is no natural transformation $\mathcal{B} \to \mathcal{O}$.

1.3 Let R be a commutative ring and let M be an R-module. Show that $(-) \otimes_R M$ is left adjoint to $\mathrm{Hom}_R(M, -)$. *Hint: Recall the universal property of the tensor product in terms of bilinear maps.*

1.4 Let $p_B \colon E \to B$ be a covering space and let $\varphi \colon B' \to B$ be continuous. Show that the morphism $p_{B'} \colon E_\varphi \to B'$ in the pullback square of p_B along φ is again a covering space. *Hint: Recall the definition of $p_B \colon E \to B$ being a covering space: for each $x \in B$ we have an open neighborhood $U \subseteq B$ of x, a discrete space D, and a homeomorphism $g \colon p_B^{-1}(U) \xrightarrow{\cong} D \times U$, which fits into the commutative triangle*



Out of this data construct corresponding triangles for $p_{B'} \colon E_\varphi \to B'$.

1.5 Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $G \colon \mathcal{D} \to \mathcal{C}$ be a functor with left adjoint $F \colon \mathcal{C} \to \mathcal{D}$. Let $D \colon I \to \mathcal{D}$ be a diagram and suppose it has the limit $\left(\lim D \xrightarrow{p_i} D(i)\right)_{i \in \mathrm{ob}\, I}$. In this exercise we will show step by step that the diagram $G \circ D$ has limit

$$\left(G(\lim D) \xrightarrow{G(p_i)} G(D(i))\right)_{i \in \mathrm{ob}\, I}. \tag{1.45}$$

(a) Show that 1.45 defines a cone on $G \circ D$.
(b) Let $\left(C \xrightarrow{f_i} G(D(i))\right)_{i \in \mathrm{ob}\, I}$ be any other cone. Show that naturality of the adjunction implies that the adjunct morphisms form a cone on D.
(c) Conclude that there is a unique arrow $F(C) \to \lim D$ factorizing the cone $F(C)$ over the universal cone $\lim D$ and show that its adjunct gives the desired unique arrow factorizing the cone C over the cone $G(\lim D)$.

Remark: We thus have shown that a right adjoint functor G is ***continuous****: It preserves all (small) universal cones and in particular we have*

$$G(\lim D) = \lim(G \circ D).$$

Dually, a left adjoint functor F is ***cocontinuous****: It preserves all (small) universal cocones and in particular we have*

$$F(\mathrm{colim}\, D) = \mathrm{colim}(F \circ D).$$

1.6 Let $G\colon \mathsf{Top} \to \mathsf{Set}$ be the forgetful functor. Show that the pullback in Top corresponds to the pullback in Set under G by constructing a left adjoint to G. Does G also have a right adjoint?

1.7 An object I in a category $\mathcal{C}$ is called **initial** if for every $X \in \mathrm{ob}(\mathcal{C})$ there exists a unique morphism $I \to X$. An object T is called **terminal** if for every $X \in \mathrm{ob}(\mathcal{C})$ there exists a unique morphism $X \to T$. An object 0 is called a **zero object** if it is both initial and terminal. A category $\mathcal{C}$ is called **pointed** if it has a zero object. For any two objects $X, Y \in \mathrm{ob}(\mathcal{C})$ in a pointed category $\mathcal{C}$, there exists a unique **zero morphism** $0_{XY}\colon X \to Y$ defined as the composition $0_{XY}\colon X \to 0 \to Y$. We define **kernel** and **cokernel** of a morphism $f\colon X \to Y$ in $\mathcal{C}$ as the equalizer and coequalizer of f and 0_{XY}, respectively.

(a) Convince yourself that in Group, Ab, K-vect, R-mod, categorical kernels and cokernels describe the usual notions. Find kernels and cokernels in $\mathsf{Top}_\bullet$.
(b) Ponder why one should not expect kernels and cokernels to exist in $\mathsf{HoTop}_\bullet$.
(c) Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of pointed categories. Show that F preserves zero objects and zero morphisms if F is left or right adjoint. Show that F preserves kernels if it is right adjoint and preserves cokernels if it is left adjoint.

1.8 A subspace $A \subseteq X$ is a **retract** of X if there exists a retraction $r\colon X \to A$. Show that retracts of Hausdorff spaces are closed. Construct a non-closed strong deformation retract $A \subseteq X$ of a T_1-space X (meaning points in X are closed). *Hint: Consider a suitable quotient space of* $I \times I$.

1.9 We saw at the end of the chapter that coequalizers can be described by pushouts. Find the dual description of equalizers in terms of pullbacks.

2 Fundamental Groupoid and van Kampen's Theorem

In many geometrically relevant situations, a space can be decomposed into smaller spaces. This might either mean that one has a suitable cover by subspaces or the space arises as a gluing of subspaces right away. Our first goal is to find means that in both cases allow to compute the fundamental group of the space in terms of the fundamental groups of its constituents. We should keep in mind, however, that the fundamental group $\pi_1(X, x_0)$ of a topological space X with base point $x_0 \in X$ depends and informs on the path component of x_0 in X only. Consequently, any decomposition theorem on the fundamental group that we endeavor to come up with will have to include assumptions on path connectedness. The proof would moreover involve some cumbersome juggling with base points. To avoid these nuisances, we generalize the concept of fundamental group to the notion of **fundamental groupoid** for which we obtain a clean statement and proof of a decomposition result: the **van Kampen theorem** in the groupoid version due to R. Brown.

Now that our journey through algebraic topology is about to really begin, let us once and for all adopt the convention that *spaces* are *topological spaces* and all occurring *maps* of spaces are meant to be *continuous maps* unless otherwise stated. The notation $X \cong Y$ shall indicate that the space X is homeomorphic to Y, whereas the notation $X \simeq Y$ shall mean that X is homotopy equivalent to Y.

2.1 The Fundamental Groupoid

A **groupoid** is a small category in which all arrows are isomorphisms. The pivotal example is the fundamental groupoid of a space.

Definition 2.1

Let X be a space. The **fundamental groupoid** is the small category $\Pi(X)$ whose objects are the points in X and whose morphisms are homotopy classes of paths relative end points. So for $x, y \in X$, we have

H. Kammeyer, *Introduction to Algebraic Topology*, Compact Textbooks in Mathematics, https://doi.org/10.1007/978-3-030-98313-0_2

$$\mathrm{Hom}_{\Pi(X)}(x, y) = \{\gamma : I \to X :\ \gamma(0) = x,\ \gamma(1) = y\}/\simeq$$

where $\gamma_1 \simeq \gamma_2$ if and only if there is $H\colon I \times I \to X$ such that $H_0 = \gamma_1$ and $H_1 = \gamma_2$ as well as $H_t(0) = x$ and $H_t(1) = y$ for all $t \in I$. Composition is given by concatenation of paths: for $x \xrightarrow{[\gamma_1]} y \xrightarrow{[\gamma_2]} z$, we set $[\gamma_2] \circ [\gamma_1] = [\gamma_1\gamma_2]$ where

$$(\gamma_1\gamma_2)(t) = \left\{ \begin{matrix} \gamma_1(2t) & \text{for } 0 \le t \le \frac{1}{2}, \\ \gamma_2(2t-1) & \text{for } \frac{1}{2} \le t \le 1 \end{matrix} \right\}.$$

Note that identity morphisms in $\Pi(X)$ are represented by constant paths and indeed, all morphisms in $\Pi(X)$ are invertible: $[\gamma]$ has inverse $[\overline{\gamma}]$, where $\overline{\gamma}$ is the reverse path of γ given by $\overline{\gamma}(t) = \gamma(1-t)$. The endomorphism set of any object in a groupoid forms a group under composition. In the case of the fundamental groupoid, we have $\mathrm{Hom}_{\Pi(X)}(x_0, x_0) = \pi_1(X, x_0)$.

Definition 2.2

A category is called **connected** if every two objects A and D can be connected by a sequence of arrows as in

$$A \longleftarrow B \longrightarrow C \longleftarrow D.$$

It is not required that there are any morphisms from A to D.

The reason for the terminology is now apparent: $\Pi(X)$ is connected if and only if X is path connected. The two fundamental groups of a path connected space with respect to two different base points are isomorphic, but not canonically so. One obtains an isomorphism after picking a relative homotopy class of paths between the base points. This phenomenon can now be formulated entirely categorically. Recall from Example 1.8 that every group G gives rise to a small category $\underline{G}$ with one object $\bullet$ and $\mathrm{Hom}_{\underline{G}}(\bullet, \bullet) = G$.

Lemma 2.3

Let $\mathcal{G}$ be a nonempty connected groupoid, and let $x \in \mathrm{ob}\,\mathcal{G}$ be any object. Then the inclusion functor

$$I_x\colon \underline{\mathrm{Aut}_{\mathcal{G}}(x)} = \underline{\mathrm{Hom}_{\mathcal{G}}(x, x)} \longrightarrow \mathcal{G}$$

is an equivalence of categories.

Proof Since $\mathcal{G}$ is small, the axiom of choice allows us to pick isomorphisms $f_y\colon x \to y$ for all $y \in \mathrm{ob}\,\mathcal{G}$ and for simplicity we choose $f_x = \mathrm{id}_x$. We define a functor $R : \mathcal{G} \to \underline{\mathrm{Aut}_{\mathcal{G}}(x)}$ on objects by $R(y) = x$ and on morphisms by:

$$R(y \xrightarrow{g} z) = x \xrightarrow{f_y} y \xrightarrow{g} z \xrightarrow{f_z^{-1}} x.$$

We have $R \circ I_x = \mathrm{id}_{\underline{\mathrm{Aut}_{\mathcal{G}(x)}}}$. It remains to show that $I_x \circ R$ is naturally isomorphic to $\mathrm{id}_{\mathcal{G}}$. We claim that the arrows f_y are actually the components of such a natural isomorphism. Indeed, let $g\colon y \to z$ be any arrow in $\mathcal{G}$. Then the diagram

$$\begin{array}{ccc} I_x \circ R(y) & \xrightarrow{I_x \circ R(g)} & I_x \circ R(z) \\ {\scriptstyle f_y}\downarrow & & \downarrow{\scriptstyle f_z} \\ \mathrm{id}(y) & \xrightarrow{\mathrm{id}(g)} & \mathrm{id}(z) \end{array}$$

in $\mathcal{G}$ is just given by:

$$\begin{array}{ccccccc} x & \xrightarrow{f_y} & y & \xrightarrow{g} & z & \xrightarrow{f_z^{-1}} & x \\ {\scriptstyle f_y}\downarrow & & & & & & \downarrow{\scriptstyle f_z} \\ y & & & \xrightarrow{\qquad g \qquad} & & & z, \end{array}$$

hence it commutes and all arrows f_y are isomorphisms. □

Applying the lemma to the fundamental groupoid gives the following result.

Corollary 2.4
Let X be a nonempty path connected space, and let $x_0 \in X$ be any base point. Then the inclusion functor

$$\underline{\pi_1(X, x_0)} \longrightarrow \Pi(X)$$

is an equivalence of categories.

In this sense, the fundamental groupoid is an enhancement of the fundamental group designed for possibly disconnected spaces.

2.2 Van Kampen's Theorem

We can now explain how the fundamental groupoid of a space can be expressed in terms of the fundamental groupoids of a suitable open cover. Let $\mathsf{Groupoid}$ be the category of groupoids, morphisms being functors. We have an inclusion functor

$$\mathsf{Group} \longrightarrow \mathsf{Groupoid}$$

which views a group G as the groupoid $\underline{G}$. It is moreover clear that the fundamental groupoid defines a functor $\Pi\colon \mathsf{Top} \to \mathsf{Groupoid}$.

Theorem 2.5 (van Kampen—Groupoid Version)
Let X be a space, and let $\mathcal{O}$ be an open cover of X, which is closed under finite intersections. Consider $\mathcal{O}$ as a small category with morphisms given by inclusions. Then restricting Π to $\mathcal{O}$ defines a diagram $\Pi|_{\mathcal{O}} : \mathcal{O} \to \mathsf{Groupoid}$ such that $\Pi(X) =$ $\operatorname{colim} \Pi|_{\mathcal{O}}$.

We can use the alternative notation $\Pi(X) = \operatorname{colim}_{U \in \mathcal{O}} \Pi(U)$ for the conclusion of the theorem to stress that $\Pi(X)$ is built up from $\Pi(U)$ for $U \in \mathcal{O}$. The inclusion functor $I_{\mathcal{O}} : \mathcal{O} \to \mathsf{Top}$ satisfies $X = \operatorname{colim} I_{\mathcal{O}}$, which we may write as $X = \operatorname{colim}_{U \in \mathcal{O}} U$. Thus the theorem says that Π has the "cocontinuity" property

$$\Pi\left(\operatorname{colim}_{U \in \mathcal{O}} U\right) = \operatorname{colim}_{U \in \mathcal{O}} \Pi(U).$$

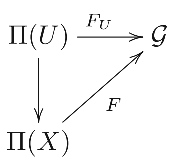
For the proof of the theorem, we need the following point-set topological consideration, which will also be convenient at a later point in the text.

Lemma 2.6 (Lebesgue)
Let X be a compact metric space with an open cover $\mathcal{U}$. Then there exists $\delta > 0$ such that every subspace $A \subseteq X$ of diameter less than δ is contained in some $U \in \mathcal{U}$.

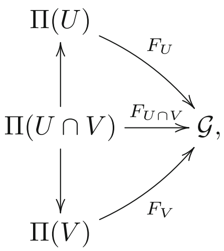
Such a constant δ is called a **Lebesgue-δ** of the cover $\mathcal{U}$.

Proof We cover $X = \bigcup_{x \in X} B_{\varepsilon(x)}(x)$ by open balls of varying radius $\varepsilon(x)$ around x such that for each $x \in X$ the ball $B_{2\varepsilon(x)}(x)$ is contained in some $U_i \in \mathcal{U}$. By compactness of X, this cover has a finite subcover $X = \bigcup_{i=1}^{n} B_{\varepsilon(x_i)}(x_i)$ and we set $\delta = \min\{\varepsilon(x_1), \ldots, \varepsilon(x_n)\}$. Now suppose there was a ball $B \subseteq X$ of radius δ, which was not contained in any of the balls $B_{2\varepsilon(x_1)}, \ldots, B_{2\varepsilon(x_n)}$. Then the midpoint of B would lie outside $\bigcup_{i=1}^{n} B_{\varepsilon(x_i)}(x_i) = X$, which is absurd. □

Proof of Theorem 2.5 Clearly, the inclusions $(\Pi(U) \to \Pi(X))_{U \in \mathcal{O}}$ define a cocone on $\Pi|_{\mathcal{O}}$. To see it is universal, let $(F_U : \Pi(U) \to \mathcal{G})_{U \in \mathcal{O}}$ be any other cocone. We have to show that there exists a unique functor $\Pi(X) \xrightarrow{F} \mathcal{G}$ such that the diagram



commutes for all $U \in \mathcal{O}$. We define F on objects by $F(x) = F_U(x)$ for any $U \in \mathcal{O}$ with $x \in U$. To see that this is well-defined, note that for any other $V \in \mathcal{O}$ with $x \in V$, we have the diagram



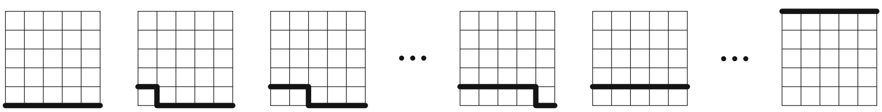
thus $F_U(x) = F_{U\cap V}(x) = F_V(x)$. To define F on morphisms, let $[\gamma\colon x \to y] \in \mathrm{Hom}_{\Pi(X)}(x, y)$ be a relative homotopy class represented by a path $\gamma\colon I \to X$ from x to y. We choose a Lebesgue-δ for the open cover $\{\gamma^{-1}(U)\}_{U\in\mathcal{O}}$ of the compact metric space $I = [0, 1]$. Subdividing $[0, 1]$ into n subintervals of length less than δ, we see that $\gamma = \gamma_1 \cdots \gamma_n$ is the concatenation of n paths γ_i each of which is contained entirely in some $U_i \in \mathcal{O}$. Set $F([\gamma]) = F_{U_n}([\gamma_n]) \circ \cdots \circ F_{U_1}([\gamma_1])$. To see that this construction is well-defined, we have to check it is independent of:

(i) The chosen subdivision of γ into $\gamma_1, \ldots, \gamma_n$ and the chosen sets $U_1, \ldots, U_n$
(ii) The representative in the relative homotopy class $[\gamma]$

(i). Any two subdivisions have a common refinement. It is therefore enough to show that $F([\gamma])$ is unaltered when refining the subdivision and possibly replacing the chosen sets U_i. To see that, it is in turn enough to discuss the case that $\gamma = \gamma_1\gamma_2$ with the images of γ_1, γ_2, and γ contained in U_1, U_2, and U_0, respectively. Using the commutativity of the cocone diagram $(F_U\colon \Pi(U) \to \mathcal{G})_{U\in\mathcal{O}}$ and the functoriality of F_{U_0}, we compute

$$F_{U_2}([\gamma_2]) \circ F_{U_1}([\gamma_1]) = F_{U_2\cap U_0}([\gamma_2]) \circ F_{U_1\cap U_0}([\gamma_1]) = F_{U_0}([\gamma_2]) \circ F_{U_0}([\gamma_1]) =$$
$$= F_{U_0}([\gamma_1\gamma_2]) = F_{U_0}([\gamma]).$$

(ii). Let $H\colon I \times I \to X$ be a homotopy relative end points from γ to γ'. So H restricts to γ on the bottom edge and to γ' on the top edge of the square $I \times I$, while it restricts to the constant map with value x on the left edge and to the constant map with value y on the right edge. Pick a Lebesgue-δ of the open cover $\{H^{-1}(U)\}_{U\in\mathcal{O}}$ of $I \times I$. We subdivide the square $I \times I$ into little squares of diameter less than δ. Then one by one we can move the path γ to γ' by homotopies relative end points through little squares as indicated in the following picture.



In doing so, when moving through left most little squares, we can move the initial point vertically upwards during the homotopy and still obtain a homotopy relative end points. Similarly, when moving through right most little squares, we can move the end point vertically upwards during the homotopy. From one step to the next, two paths γ_1 and γ_2 differ only by a little square, so γ_i is a concatenation $\gamma_i = \gamma_{\text{initial}}\gamma^i_{\text{middle}}\gamma_{\text{end}}$ with $i = 1, 2$. Using the same sets in $\mathcal{O}$ for the subdivisions of γ_1 and γ_2 induced by the little squares, it follows that

$$F([\gamma_1]) = \cdots \circ F_U([\gamma^1_{\text{middle}}]) \circ \cdots = \cdots \circ F_U([\gamma^2_{\text{middle}}]) \circ \cdots = F([\gamma_2])$$

because $\gamma^1_{\text{middle}} \simeq \gamma^2_{\text{middle}}$ relative end points. We conclude $F([\gamma]) = F([\gamma'])$.

By construction, F is a functor that factorizes the cocone $(\Pi(U) \stackrel{F_U}{\to} \mathcal{G})_{U\in\mathcal{O}}$ over the cocone $(\Pi(U) \to \Pi(X))_{U\in\mathcal{O}}$ and it is unique with this property. □

The groupoid version of van Kampen's theorem has an elegant statement and a slick proof. It is however not apparent how it can be employed for actual computations of fundamental groups. Therefore, we will now make the transition back from groupoids to groups and derive the group version of van Kampen's theorem from the groupoid version.

Theorem 2.7 (van Kampen—Group Version)
Let X be a nonempty space, and let $\mathcal{O}$ be a cover by open path connected subsets that all contain a given point $x_0 \in X$. Suppose in addition that $\mathcal{O}$ is closed under finite intersections. Consider $\mathcal{O}$ as a small category with morphisms given by pointed inclusions. Then the diagram $\pi_1|_{\mathcal{O}} \colon \mathcal{O} \to \mathsf{Group}$ satisfies $\pi_1(X, x_0) = \operatorname{colim} \pi_1|_{\mathcal{O}}$.

Similarly as before, the inclusion functor $I_{\mathcal{O}} \colon \mathcal{O} \to \mathsf{Top}_\bullet$ is a diagram with $(X, x_0) = \operatorname{colim} I_{\mathcal{O}}$ so that we can restate the assertion of the theorem as

$$\pi_1(\operatorname{colim}_{U\in\mathcal{O}}(U, x_0)) = \operatorname{colim}_{U\in\mathcal{O}} \pi_1(U, x_0).$$

Proof Since every point in X can be joined by a path to x_0 within some $U \in \mathcal{O}$, we see that X is path connected. It follows from Corollary 2.4 that the inclusion functor $I \colon \underline{\pi_1(X, x_0)} \to \Pi(X)$ is an equivalence of categories. The proof of Corollary 2.4 also reveals that we obtain the inverse functor $R \colon \Pi(X) \to \underline{\pi_1(X, x_0)}$ by picking paths $\gamma_y \colon x_0 \to y$ for each $y \in X$ and setting

$$R([\gamma \colon y \to z]) = [x_0 \xrightarrow{\gamma_y} y \xrightarrow{\gamma} z \xrightarrow{\gamma_z^{-1}} x_0].$$

Let us first assume that the cover $\mathcal{O}$ is finite. Then we can pick the paths γ_y cleverly within $\bigcap_{U\ni y} U$ and again we agree that γ_{x_0} is the constant path. This choice has the effect that we also obtain inverse functors $R_U \colon \Pi(U) \to \underline{\pi_1(U, x_0)}$ and these satisfy the commutativity

$$\begin{array}{ccc} \Pi(U) & \xrightarrow{R_U} & \underline{\pi_1(U,x_0)} \\ \downarrow & & \downarrow \\ \Pi(V) & \xrightarrow{R_V} & \underline{\pi_1(V,x_0)}. \end{array}$$

Now let $(f_U : \pi_1(U,x_0) \to G)_{U\in\mathcal{O}}$ be any cocone on the diagram $\pi_1|_{\mathcal{O}}$. We obtain an induced cocone $(\underline{f_U} : \underline{\pi_1(U,x_0)} \to \underline{G})_{U\in\mathcal{O}}$ on the diagram of groupoids $\underline{\pi_1}|_{\mathcal{O}}$. The above commutative squares express that the compositions

$$\left(\Pi(U) \xrightarrow{R_U} \underline{\pi_1(U,x_0)} \xrightarrow{\underline{f_U}} \underline{G}\right)_{U\in\mathcal{O}} \tag{2.8}$$

form a cocone on the diagram $\Pi|_{\mathcal{O}}$. By the groupoid version of van Kampen's theorem, there exists a unique functor $F : \Pi(X) \to \underline{G}$ such that the diagram

$$\begin{array}{ccccc} \Pi(U) & \xrightarrow{R(U)} & \underline{\pi_1(U,x_0)} & \xrightarrow{\underline{f_U}} & \underline{G} \\ \downarrow & & & \nearrow_{F} & \\ \Pi(X) & & & & \end{array}$$

commutes for all $U \in \mathcal{O}$. We extend this diagram on the left by inclusion functors.

$$\begin{array}{ccccccc} \underline{\pi_1(U,x_0)} & \xrightarrow{I_U} & \Pi(U) & \xrightarrow{R(U)} & \underline{\pi_1(U,x_0)} & \xrightarrow{\underline{f_U}} & \underline{G} \\ \downarrow & & \downarrow & & & \nearrow_{F} & \\ \underline{\pi_1(X,x_0)} & \xrightarrow{I} & \Pi(X) & & & & \end{array}$$

Since we have $R_U \circ I_U = \mathrm{id}_{\underline{\pi_1(U,x_0)}}$, the morphism $\underline{f} := F \circ I$ is simply a group homomorphism $f : \pi_1(X,x_0) \to G$ that satisfies $f_U = f \circ i_U$ for the inclusion $i_U : \pi_1(U,x_0) \to \pi_1(X,x_0)$ as required. To see uniqueness, let $g : \pi_1(U,x_0) \to G$ be another morphism satisfying $f_U = g \circ i_U$ for all $U \in \mathcal{O}$. Then we have the diagram

$$\begin{array}{ccccc} \Pi(U) & \xrightarrow{R_U} & \underline{\pi_1(U,x_0)} & \xrightarrow{\underline{f_U}} & \underline{G} \\ \downarrow & & \downarrow{\scriptstyle \underline{i_U}} & \nearrow_{\underline{g}} & \\ \Pi(X) & \xrightarrow[R]{} & \underline{\pi_1(X,x_0)} & & \end{array}$$

in Groupoid. This shows that the morphism $\underline{g} \circ R$ factorizes the cocone (2.8) over the cocone $(\Pi(U) \to \Pi(X))_{U \in \mathcal{U}}$. The latter cocone is universal by the groupoid version of van Kampen's theorem, so we have $\underline{g} \circ R = F$ by uniqueness. Precomposing both sides of this equality with I, we get $\underline{g} = F \circ I = \underline{f}$, hence $g = f$.

Let us now assume that $\mathcal{O}$ is a possibly infinite cover and again let the family $(f_U : \pi_1(U, x_0) \to G)_{U \in \mathcal{O}}$ be a cocone on $\pi_1|_{\mathcal{O}}$. For every finite intersection stable subcover $\mathcal{S} \subseteq \mathcal{O}$, let $U_{\mathcal{S}} = \bigcup_{U \in \mathcal{S}} U$. The restricted cocone $(f_U : \pi_1(U, x_0) \to G)_{U \in \mathcal{S}}$ defines a unique factorization map $F_{\mathcal{S}} : \pi_1(U_{\mathcal{S}}, x_0) \to G$ by what we have proven so far. By compactness of I and $I \times I$, any given loop γ and any homotopy H in X lies in a suitable $U_{\mathcal{S}}$ as above. From this, it follows that setting $F([\gamma]) = F_{\mathcal{S}}([\gamma])$ gives the well-defined and unique required factorization map $F : \pi_1(X, x_0) \to G$ of the cocone $(f_U : \pi_1(U, x_0) \to G)_{U \in \mathcal{O}}$. □

Van Kampen's theorem is most commonly applied in the following situation. We pick a base point $x_0 \in X$ in a nonempty space, and we decompose $X = U_1 \cup U_2$ with U_1, U_2 open and $U_1, U_2, U_1 \cap U_2$ path connected such that $x_0 \in U_1 \cap U_2$. In that case, the group version says that

$$\begin{array}{ccc} \pi_1(U_1 \cap U_2, x_0) & \longrightarrow & \pi_1(U_1, x_0) \\ \downarrow & & \downarrow \\ \pi_1(U_2, x_0) & \longrightarrow & \pi_1(X, x_0) \end{array}$$

is a pushout square in Group.

Example 2.9 For the n-dimensional sphere S^n with $n \geq 2$, we choose a base point $x_0 \in S^n$ on the equator and define U_1 as the complement of the south pole and U_2 as the complement of the north pole. Then U_1 and U_2 are contractible so that the pushout square looks like

$$\begin{array}{ccc} \pi_1(U_1 \cap U_2, x_0) & \longrightarrow & \{1\} \\ \downarrow & & \downarrow \\ \{1\} & \longrightarrow & \pi_1(S^n, x_0). \end{array}$$

It follows that S^n is simply connected for $n \geq 2$.

Example 2.10 Consider the figure eight $S^1 \vee S^1$, and let U_1, U_2 consist of one loop each together with a little open overlap into the other loop. Then the intersection $U_1 \cap U_2$ looks like the letter "X," hence $U_1 \cap U_2 \simeq \bullet$. So the pushout square

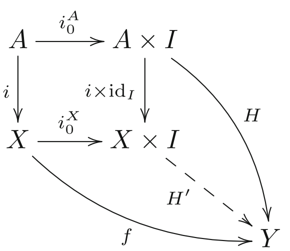
$$\begin{array}{ccc} \{1\} & \longrightarrow & \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{Z} & \longrightarrow & \pi_1(S^1 \vee S^1, \bullet) \end{array}$$

reduces in fact to a coproduct diagram and $\pi_1(S^1 \vee S^1, \bullet) \cong \mathbb{Z} * \mathbb{Z} \cong \mathcal{F}(\{a, b\}) =: F_2$ is the free group on two letters a and b from Example 1.15. The letters a and b correspond to loops that wind around one of the circles once. Inductively, the fundamental group of the one point union of n copies of the circle is isomorphic to F_n, the free group on n letters. More generally, for any (possibly infinite) index set I, the full version of van Kampen's theorem shows that $\pi_1(\bigvee_{i \in I} S^1, \bullet) \cong \mathcal{F}(I)$ is the free group on the alphabet I.

In the last example, the overlaps of the subsets U_1 and U_2 are somewhat irritating and it would have felt more natural to decompose the space $S^1 \vee S^1$ just into the two circles. Similarly, in the preceding example, it would have been more efficient to subdivide the sphere S^n into the closed upper hemisphere and the closed lower hemisphere. The intersections would then be closed subspaces, namely the point $\bullet$ for $S^1 \vee S^1$ and the equatorial S^{n-1} for S^n. However, we had to require that the subspaces U_i have some overlap to make sure we work with an open cover of the space so that the assumptions of van Kampen's theorem are met. But in the end, it does not matter how these thickenings actually look like. It only matters that one can choose an open neighborhood that contains the closed subspace of interest as deformation retract. Subspaces with this property will be of increasing interest as one advances through algebraic topology, which is why we will dedicate the next section to their study. Afterwards, we can give another version of van Kampen's theorem that is most suitable for applications.

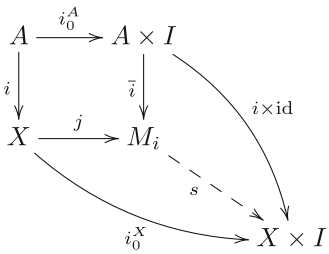
2.3 Cofibrations and Homotopy Pushouts

A map $i: A \to X$ of spaces has the **homotopy extension property** (HEP) for a space Y if for each homotopy $H: A \times I \to Y$ and for each map $f: X \to Y$ with $f(i(a)) = H(a, 0)$ for all $a \in A$, there exists a homotopy $H': X \times I \to Y$ such that $H'(i(a), t) = H(a, t)$ and $H'(x, 0) = f(x)$ for all $a \in A$, $x \in X$, and $t \in I$. The homotopy H' is called an **extension** of H with **initial condition** f. A map $i: A \to X$ is called a **cofibration** if it has the HEP for all spaces Y. Setting $i_0^A(a) = (a, 0)$ and $i_0^X(x) = (x, 0)$, we can express the definition by the diagram



(2.11)

in which we require *only existence* of H', *not uniqueness*. The definition of cofibration is of course impractical to verify directly. Therefore it is good to know that if i has the HEP for its own mapping cylinder M_i, then i has the HEP for all spaces. To prove this, we first observe that the pushout diagram

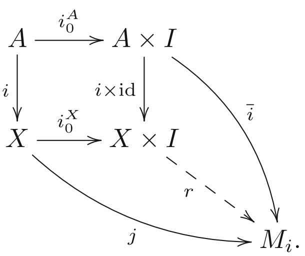


uniquely defines the map $s\colon M_i \to X \times I$. If i is the inclusion of a subspace, then s is a continuous bijection onto the image $X \times \{0\} \cup A \times I$. But the topology of M_i might be finer than the subspace topology within $X \times I$. However, we see that s is a homeomorphism onto the image, hence a subspace inclusion, if i and thus also $i \times \mathrm{id}$ is the inclusion of a *closed* subspace. By Proposition 2.12 (iii) below, s is also a homeomorphism onto the image if i is a cofibration.

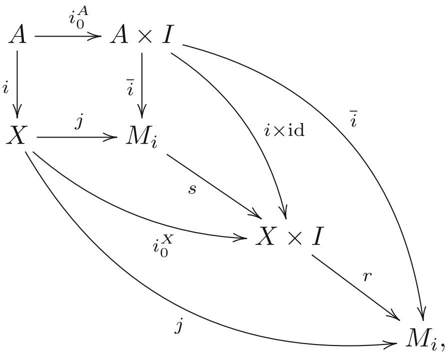
Proposition 2.12
Consider a map $i\colon A \to X$. The following are equivalent:

(i) The map i is a cofibration.
(ii) The map i has the HEP for M_i.
(iii) The map s has a retraction $r\colon X \times I \to M_i$ so that $r \circ s = \mathrm{id}_{M_i}$.

Proof (i) ⇒ (ii). Trivial. (ii) ⇒ (iii). By the HEP of i for M_i, we obtain the (non-unique) arrow $r\colon X \times I \to M_i$ from the diagram

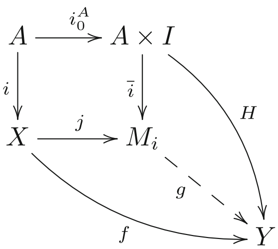


The defining diagrams of s and r unify to the diagram

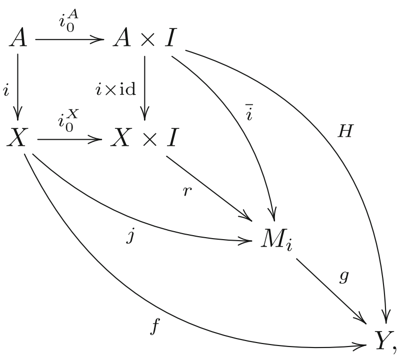


hence $r \circ s = \mathrm{id}_{M_i}$ follows from the uniqueness statement in the universal property of the pushout defining M_i.

(iii) $\Rightarrow$ (i). This is formally similar, swapping HEP and pushout diagrams. Given $H: A \times I \to Y$ and $f: X \to Y$ with $f \circ i = H \circ i_0^A$, we obtain the pushout



which unifies with the defining diagram of r to the diagram



hence $g \circ r$ provides an extension of H with initial condition f. □

It is immediate from the definition that the composition of two cofibrations is a cofibration. Not quite immediate but very helpful for the right intuition is the following lemma.

Lemma 2.13
If $i: A \to X$ is a cofibration, then it is the inclusion of a subspace. If in addition X is Hausdorff, then $i: A \subseteq X$ is closed.

Proof Let $i_1^X: X \to X \times I$ be the inclusion $i_1^X(x) = (x, 1)$. For the end map $\bar{i}_1$ of the homotopy $\bar{i}: A \times I \to M_i$ in the mapping torus M_i of i, we have

$$\bar{i}_1(a) = \bar{i}(a, 1) = r(s(\bar{i}(a, 1))) = r \circ (i \times \mathrm{id})(a, 1) = r(i(a), 1) = r \circ i_1^X \circ i(a) \tag{2.14}$$

for all $a \in A$. We already saw in Example 1.38 that $\bar{i}_1$ is a homeomorphism onto the image, so $\bar{i}_1^{-1} \circ r \circ i_1^X|_{\mathrm{im}\, i}$ is the continuous inverse of $i: A \to i(A)$. This proves the first part of the lemma. More than that, we get that

$$A \xrightarrow{\;i\;} X \underset{i_1^X}{\overset{s \circ r \circ i_1^X}{\rightrightarrows}} X \times I$$

is an equalizer. Indeed, applying s to the equation in (2.14) shows that the parallel arrows agree on $\mathrm{im}\, i$. Conversely, if $x \in X$ satisfies $i_1^X(x) = s \circ r \circ i_1^X(x)$, then $(x, 1) \in \mathrm{im}\, s$. As the second coordinate is one, we must have in fact $(x, 1) \in \mathrm{im}(s \circ \bar{i})$ so $x \in \mathrm{im}\, i$. Now the second part of the lemma follows from Example 1.29. □

Thus it comes with no loss of generality to always view cofibrations as pairs (X, A) with the HEP for all spaces. This justifies the word "extension" in the HEP. One can moreover show that equalizers of continuous maps of **compactly generated weakly Hausdorff spaces** are closed [24, Proposition 2.15]. So if X is compactly generated weakly Hausdorff, one can still conclude that the subspace A in a cofibration (X, A) is closed. The upshot is that non-closed cofibrations can safely be considered pathological. We will now give a constructive characterization of closed cofibrations.

Theorem 2.15
Let (X, A) be a pair of spaces. The following are equivalent:

(i) The pair (X, A) is a closed cofibration.
(ii) There exists a retraction $R: X \times I \longrightarrow X \times \{0\} \cup A \times I$ and $A \subseteq X$ is closed.

(continued)

(iii) There exists a map $u\colon X \to I$ and a homotopy $h\colon X \times I \to X$ such that:
(1) $u^{-1}(0) = A$.
(2) $h(x, 0) = x$ for all $x \in X$.
(3) $h(a, t) = a$ for all $a \in A$ and $t \in I$.
(4) $h(x, 1) \in A$ for all $x \in X$ with $u(x) < 1$.

So the map u in (iii) describes the closed subset $A = u^{-1}(0)$ as a strong deformation retract of the open neighborhood $U = u^{-1}([0, 1))$. In view of this characterization, a closed cofibration (X, A) is also called a **neighborhood deformation retract** or for short an **NDR pair** or just an **NDR**.

Proof (i) ⇒ (ii). Setting $Y = X \times \{0\} \cup A \times I$, $f(x) = (x, 0)$, and $H(a, t) = (a, t)$ in the HEP (2.11), we obtain a retraction $R\colon X \times I \longrightarrow X \times \{0\} \cup A \times I$.

(ii) ⇒ (i). Let $H\colon A \times I \to Y$ be a homotopy, and let $f\colon X \to Y$ be an initial condition. As we observed above Proposition 2.12,

$$\begin{array}{ccc} A & \xrightarrow{i_0^A} & A \times I \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i \times \mathrm{id}} \\ X & \xrightarrow{i_0^X} & X \times \{0\} \cup A \times I \end{array}$$

is a pushout square because $A \subseteq X$ is closed. By the universal property, H and f define a map $F\colon X \times \{0\} \cup A \times I \to Y$ and $H' = F \circ R$ is an extension of H with initial condition f.

(ii) ⇒ (iii). Define $h\colon X \times I \to X$ by $h = \mathrm{pr}_X \circ R$. Then (iii) and (iii) are clear. Define $u\colon X \to I$ by $u(x) = \max_{t \in I} |t - \mathrm{pr}_I(R(x, t))|$. Then clearly $A \subseteq u^{-1}(0)$. If conversely $u(x) = 0$, then $\mathrm{pr}_X(R(x, (0, 1])) \subseteq A$. Since A is closed, this shows that also $\mathrm{pr}_X(R(x, 0)) \in A$ by continuity, so we have verified (iii). To see (iii), suppose $u(x) < 1$. Then in particular $|1 - \mathrm{pr}_I(R(x, 1))| < 1$, which shows $R(x, 1) = (a, t)$ for some $a \in A$ and some $t > 0$. Therefore $h(x, 1) = \mathrm{pr}_X(R(x, 1)) = \mathrm{pr}_X(a, t) = a \in A$, which gives (iii). It remains to show that u is continuous. To this end, we consider the continuous functions $d\colon X \times I \to I$ given by $d(x, t) = |t - \mathrm{pr}_I(R(x, t))|$ and $d_t\colon X \to I$ given by $d_t(x) = d(x, t)$. Then for $s \in I$, both the sets

$$u^{-1}([0, s]) = \bigcap_{t \in I} d_t^{-1}([0, s]) \quad \text{and} \quad u^{-1}([s, 1]) = \mathrm{pr}_X(d^{-1}([s, 1]))$$

are closed in X, the latter by Proposition A.2 (i). The complements $(s, 1]$ and $[0, s)$ for $s \in I$ form a subbasis of the topology of I, so u is continuous.

(iii) ⇒ (ii). By continuity of u and (iii), the subset $A \subset X$ is closed. We define a map $R\colon X \times I \longrightarrow X \times \{0\} \cup A \times I$ by:

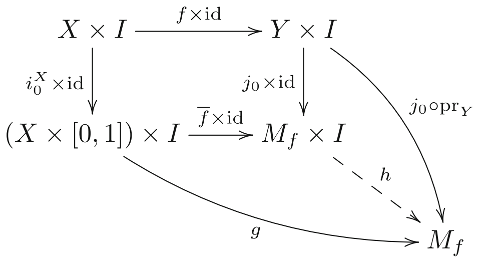
$$R(x, t) = (h(x, \tfrac{t}{u(x)}), 0) \text{ if } u(x) > t \text{ and } R(x, t) = (h(x, 1), t - u(x)) \text{ if } u(x) \le t.$$

Continuity of R only needs proof on the subset $A \times \{0\} \subseteq X \times I$. Let $U \subseteq X$ be a neighborhood of $a \in A$. As we have $h(a, t) = a$ for all $t \in I$, continuity of h and compactness of I imply that there exists a neighborhood $V \subseteq X$ of a such that $h(V \times I) \subseteq U$. Hence for $t > 0$, we have $R(V \times [0, t)) \subseteq U \times [0, t]$, which shows R is continuous at $(a, 0)$. By construction, R is a retraction: it fixes all points in $X \times \{0\} \cup A \times I$. □

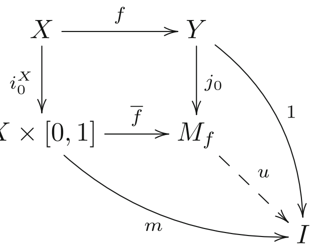
Remark 2.16
Assertions (i) and (ii) in the theorem remain equivalent if one drops the closedness condition from both statements. The proof requires a tricky point-set topological argument that was found by Strøm [25, Theorem 2].

Remark 2.17
If we have a retraction $R\colon X \times I \longrightarrow X \times \{0\} \cup A \times I$ as occurring in the theorem, then $X \times \{0\} \cup A \times I$ is automatically a strong deformation retract of $X \times I$. Indeed, setting $R(x, t) = (R_1(x, t), R_2(x, t))$, a strong deformation retraction $H\colon X \times I \times I \longrightarrow X \times I$ is given by $H(x, t, s) = (R_1(x, t(1 - s)), st + (1 - s)R_2(x, t))$.

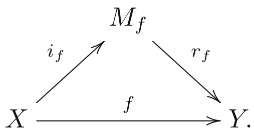
Most notably, for every map of spaces $f\colon X \to Y$, the inclusion $i_f\colon X \to M_f$ into the mapping cylinder constructed in Example 1.38 is a closed cofibration. This can be inferred from the NDR characterization in the theorem because with the help of Proposition 1.41, we obtain the required homotopy h from the pushout



with $g((x, t), s) = \overline{f}(x, \min\{(1 + s)t, 1\})$ and the required map u from the pushout



where $m(x, t) = \min\{2(1-t), 1\}$. We can now repeat our remark from Example 1.38 in the more informed manner that in the category Top, every arrow is the composition of a cofibration and a homotopy equivalence. Even better, every arrow is the composition of a closed cofibration and a strong deformation retraction.



Example 2.18 It is probably fair to say that the pair (D^n, S^{n-1}) is the single most important example of a closed cofibration. To see the NDR property, just notice that the inclusion $S^{n-1} \to D^n$ can be identified with the mapping cylinder inclusion i_f for the unique map $f: S^{n-1} \to \bullet$. More generally, for every space X, the pair (CX, X) is an NDR because the base inclusion $X \subset CX$ is the mapping cylinder inclusion i_f for $f: X \to \bullet$.

A cofibration (X, A) is called **trivial** or **acyclic** if the inclusion $i: A \subseteq X$ is a homotopy equivalence. It turns out that then A is a strong deformation retract of X. So a trivial closed cofibration can also be described as a **DR pair**, meaning an NDR pair for which the function u can be chosen with $u(x) < 1$ for all $x \in X$.

Proposition 2.19
Let (X, A) be a trivial cofibration. Then A is a strong deformation retract of X.

Proof By assumption, the inclusion $i: A \to X$ has a homotopy inverse $r: X \to A$ so that $r \circ i \simeq_H \mathrm{id}_A$ and $i \circ r \simeq_G \mathrm{id}_X$. Using r as initial condition, the HEP of i for A gives an extension $H': X \times I \to A$ of $H: A \times I \to A$. We set $r' := H'_1$, hence $r' \circ i = \mathrm{id}_A$ and $i \circ r' \simeq_{i \circ H'_{1-t}} i \circ r \simeq_G \mathrm{id}_X$. So the concatenation G' of $i \circ H'_{1-t}$ and G is a deformation retraction $i \circ r' \simeq_{G'} \mathrm{id}_X$. It remains to turn G' into a homotopy that fixes A pointwise throughout. To this end, we define a homotopy of homotopies $g: A \times I \times I \to X$ by:

$$g(a, t, s) = \left\{ \begin{array}{ll} G'(a, 1-2t) & \text{for } t \in [0, \frac{1}{2} - \frac{s}{2}], \\ G'(a, 2t-1) & \text{for } t \in [\frac{1}{2} + \frac{s}{2}, 1], \\ G'(a, s) & \text{otherwise} \end{array} \right\}.$$

The homotopy g is well-defined and continuous because the two line segments $\{(t, s) \in I \times I : s = \pm(1 - 2t)\}$ where the definitions overlap are closed subsets of the square $I \times I$. On the edge $\{(t, s) \in I \times I : s = 0\}$ of the square $I \times I$, the map g is the restriction to $A \times I$ of the homotopy $G'': X \times I \to X$ given by:

$$G''(x, t) = \left\{ \begin{array}{ll} G'(i(r'(x)), 1-2t) & \text{for } t \in [0, \frac{1}{2}], \\ G'(x, 2t-1) & \text{for } t \in [\frac{1}{2}, 1] \end{array} \right\}.$$

which is well-defined and continuous because for $t = \frac{1}{2}$ we have

$$G'(i(r'(x)), 0) = i(r'(i(r'(x)))) = i(r'(x)) = G'(x, 0).$$

On all points of the other three edges, g restricts to i. The pair $(X \times I, A \times I)$ is a cofibration by the criterion in Proposition 2.12 (iii) because Proposition 1.41 gives a homeomorphism $M_{i \times \mathrm{id}_I} \cong M_i \times I$. Hence we can extend the homotopy $g\colon (A \times I) \times I \to X$ to a homotopy $g'\colon (X \times I) \times I \to X$ with initial condition $g'(x, t, 0) = G''(x, t)$. Following g' along the three edges from $(0, 0)$ to $(0, 1)$ to $(1, 1)$ to $(1, 0)$ gives a homotopy from $i \circ r'$ to id_X fixing A pointwise at all times. □

Corollary 2.20
The map $f\colon X \to Y$ is a homotopy equivalence if and only if $i_f\colon X \to M_f$ is the inclusion of a strong deformation retract.

Proof Consider the factorization $f = r_f \circ i_f$ in HoTop. We see that f is a homotopy equivalence if i_f is. Conversely, if f is a homotopy equivalence, then the cofibration i_f is trivial, hence it is the inclusion of a strong deformation retract. □

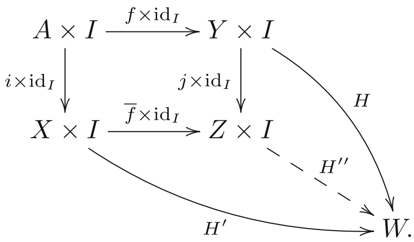
The converse of Proposition 2.19 is wrong by Exercise 2.2. Cofibrations behave nicely with respect to pushouts. This is the content of the next two theorems.

Theorem 2.21
Consider a pushout square in Top

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow{\overline{f}} & Z. \end{array}$$

(i) If i is a cofibration, then j is a cofibration.
(ii) If i is an NDR, then j is an NDR.

Proof Let $H\colon Y \times I \to W$ be a homotopy, and let $g\colon Z \to W$ be an initial condition so that $g \circ j = H_0$. From the HEP of i, we obtain a homotopy $H'\colon X \times I \to W$ extending $H \circ (f \times \mathrm{id}_I)$ with initial condition $g \circ \overline{f}$. Hence H and H' form a cocone on the diagram consisting of $f \times \mathrm{id}_I$ and $i \times \mathrm{id}_I$ so that from Proposition 1.41, we get a homotopy $H''\colon Z \times I \to W$ as the unique dashed arrow in the diagram

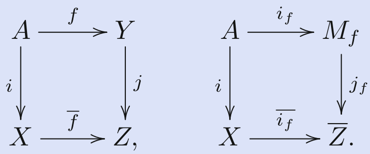


By the upper triangle, H'' is an extension of H. Moreover, for $z \in Z \setminus \operatorname{im} j$, there exists $x \in X$ with $\overline{f}(x) = z$ and by the lower triangle we have $H_0''(z) = H_0'(x) = g(\overline{f}(x)) = g(z)$. So H'' extends H with initial condition g, which shows (i). Part (ii) is just the combination of part (i), Theorem 2.15, and Theorem 1.42 (ii). □

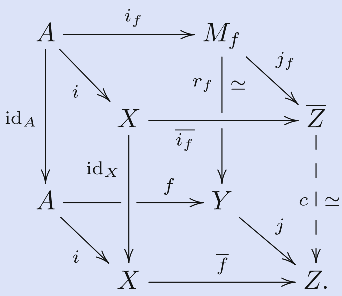
Theorem 2.22
Let $i: A \to X$ be a cofibration, let $f: A \to Y$ be a map, and let

$$A \xrightarrow{i_f} M_f \xrightarrow{r_f} Y$$

be the decomposition of f into cofibration and homotopy equivalence. Consider the two pushout squares



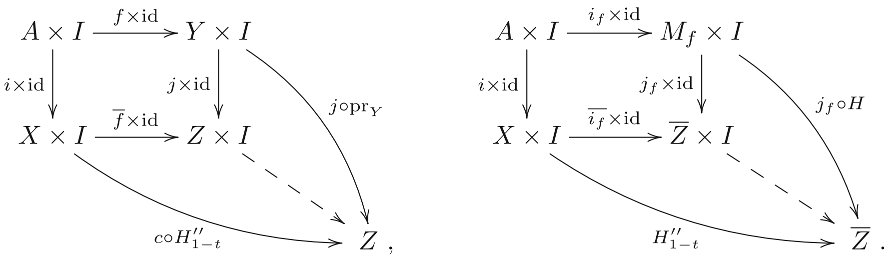
There exists a unique homotopy equivalence $c: \overline{Z} \xrightarrow{\simeq} Z$ completing the diagram



Proof The maps $j \circ r_f$ and $\overline{f}$ form a cocone on the upper pushout so that the universal property gives a unique map $c: \overline{Z} \to Z$ such that the right and the front cube face commute. It remains to see that c is a homotopy equivalence. To find the homotopy inverse, we would like to use the canonical inclusion $j_0 \colon Y \to M_f$ from Example 1.38 (where $X = A$). The idea is to apply the universal property of the lower pushout to the cocone formed by $j_f \circ j_0$ and $\overline{i_f}$. The problem is that these maps do not form a cocone in Top: we do not have $j_f \circ j_0 \circ f = \overline{i_f} \circ i$. Nonetheless, they do form a cocone in HoTop: we have $j_f \circ j_0 \circ f \simeq_{H'} \overline{i_f} \circ i$. The homotopy H' is obtained from the deformation retraction $j_0 \circ r_f \simeq_H \mathrm{id}_{M_f}$ via $H' = j_f \circ H \circ (i_f \times \mathrm{id}_I)$. So visually, H' slides A along the cylindrical part of $\overline{Z}$ from Y toward X. By the HEP of i, the reverse homotopy H'_{1-t} extends to a homotopy $H'' \colon X \times I \to \overline{Z}$ with $H''_0 = \overline{i_f}$. Let $e = H''_1$ be the end map of the homotopy. We now have

$$e \circ i = H''_1 \circ i = H'_0 = j_f \circ j_0 \circ f,$$

so e and $j_f \circ j_0$ are a cocone in Top on the diagram consisting of i and f. We obtain the map $d \colon Z \to \overline{Z}$ from the universal property of the lower pushout. Finally, using again Proposition 1.41, we find the homotopies $c \circ d \simeq \mathrm{id}_Z$ and $d \circ c \simeq \mathrm{id}_{\overline{Z}}$ as the unique dashed arrows in the pushout diagrams



□

Given two maps $X \xleftarrow{f_2} A \xrightarrow{f_1} Y$ in Top, the pushout of $X \xleftarrow{f_2} A \xrightarrow{i_{f_1}} M_{f_1}$ can more symmetrically be described by the pushout

$$\begin{array}{ccc} A \times I & \xrightarrow{\overline{f_2}(a,1-t)} & M_{f_2} \\ {\scriptstyle \overline{f_1}(a,t)}\downarrow & & \downarrow \\ M_{f_1} & \longrightarrow & M_{f_1,f_2} \end{array}$$

and is known as the **double mapping cylinder** of f_1 and f_2. For every square

$$\begin{array}{ccc} A & \xrightarrow{f_2} & Y \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle g_2} \\ X & \xrightarrow{g_1} & Z \end{array}$$

in Top, which is **homotopy commutative**, meaning $g_1 \circ f_1 \simeq_H g_2 \circ f_2$, the maps g_1 and $H(a, t)$ induce a map $c_1 : M_{f_1} \to Z$, while the maps g_2 and $H(a, 1-t)$ induce a map $c_2 : M_{f_2} \to Z$. Finally c_1 and c_2 give a **comparison map** $c\colon M_{f_1, f_2} \to Z$.

Definition 2.23
A homotopy commutative square

$$\begin{CD} A @>{f_2}>> Y \\ @V{f_1}VV @VV{g_2}V \\ X @>{g_1}>> Z \end{CD}$$

in Top is called a **homotopy pushout** if there exists a homotopy $g_1 \circ f_1 \simeq_H g_2 \circ f_2$ such that the comparison map $c\colon M_{f_1, f_2} \to Z$ is a homotopy equivalence.

Categorical pushouts in Top may or may not be homotopy pushouts: the pushout of the diagram $D^2 \leftarrow S^1 \rightarrow D^2$ is a homotopy pushout, whereas the pushout of $\bullet \leftarrow S^1 \rightarrow \bullet$ is not. In fact, the notion of homotopy pushout is designed to remedy the defect that categorical pushouts of homotopy equivalent spaces need not be homotopy equivalent. Theorem 2.22 says that a pushout square

$$\begin{CD} A @>{f}>> Y \\ @V{i}VV @VV{j}V \\ X @>{\overline{f}}>> Z. \end{CD}$$

in Top is a homotopy pushout if either i or f (by symmetry) is a cofibration. The unique homotopy equivalence c in the theorem is precisely the comparison map for the constant homotopy from $\overline{f} \circ i$ to $j \circ f$.

Theorem 2.24
Consider a pushout square in Top

$$\begin{CD} A @>{f}>> Y \\ @V{i}VV @VV{j}V \\ X @>{\overline{f}}>> Z. \end{CD}$$

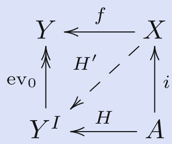
Suppose that i is a cofibration and that f is a homotopy equivalence. Then also $\overline{f}$ is a homotopy equivalence.

Proof Since i is a cofibration, we can revisit the commutative cube from Theorem 2.22. Since f is a homotopy equivalence, the map i_f is the inclusion of a strong deformation retract by Corollary 2.20. Hence $\overline{i_f}$ is the inclusion of a strong deformation retract by Theorem 1.42(iii) whence $\overline{f}$ is a homotopy equivalence by commutativity of the front face of the cube. □

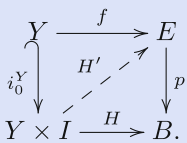
Corollary 2.25
Let (X, A) be a cofibration and assume that A is contractible. Then the collapse map $q\colon X \to X/A$ is a homotopy equivalence.

Proof Apply the theorem to the pushout (1.37). □

Remark 2.26
We still owe the reader an explanation of the strange terminology "cofibration." To demystify the word, let us first observe that the HEP of $i\colon A \to X$ for a space Y can also be expressed by the diagram



where Y^I is the space of maps $I \to Y$ with the compact-open topology and ev_0 is the evaluation map in zero, $\mathrm{ev}_0(g) = g(0)$ for $g\colon I \to Y$. An element in Y^I is a continuous choice of elements $y \in Y$, one for each $i \in I$. So we may think of the space Y^I as a "continuous product" of copies of Y parameterized by the interval I. The dual concept should thus be the "continuous coproduct" $Y \times I$ of copies of Y parameterized by I because an element $(y, t) \in Y \times I$ is a choice of a single element $y \in Y$ in the t-th copy of Y. The dual of the HEP is thus encoded by the diagram



Reflecting on it, one realizes that it asserts a **homotopy lifting property** (HLP) of the map p. The homotopy H' should be a **lift** of the homotopy H along the map p with **initial condition** f. A map $p\colon E \to B$ with the HLP for all spaces Y is called a **fibration** because the main examples of fibrations are **fiber bundles**. These are defined by a local triviality condition much like covering spaces though fibers are not required to be discrete. In fact, the statement from (1.23) that pullbacks of covering maps are covering maps extends to the statement that pullbacks of fibrations are fibrations. As

(continued)

such, it is dual to Theorem 2.21 (i) stating that pushouts of cofibrations are cofibrations. In a similar vein, not only is every arrow in Top a composition of a cofibration and a homotopy equivalence, it is also a composition of a homotopy equivalence and a fibration. These facts form a key point in modern axiomatic approaches to homotopy theory built on so-called model categories. Duality statements as in this remark are loosely subsumed under the term **Eckmann–Hilton duality**.

2.4 Computing Fundamental Groups

As a reward for the hard work of the previous section, we obtain the following version of van Kampen's theorem, which is a powerful tool to carry out actual computations of fundamental groups.

Theorem 2.27 (van Kampen—Pushout Version)
Let

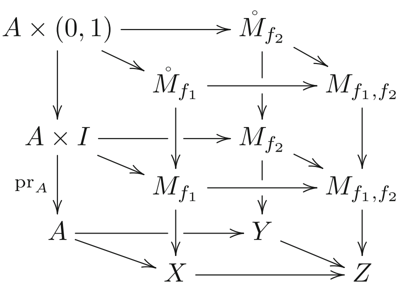
$$\begin{array}{ccc} A & \xrightarrow{f_2} & Y \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle g_2} \\ X & \xrightarrow{g_1} & Z. \end{array}$$

be a pushout of nonempty path connected spaces such that f_1 or f_2 is a cofibration. Pick $a_0 \in A$ and set $x_0 = f_1(a_0)$, $y_0 = f_2(a_0)$, $z_0 = g_1(f_1(a_o)) = g_2(f_2(a_0))$. Then

$$\begin{array}{ccc} \pi_1(A, a_0) & \xrightarrow{\pi_1(f_2)} & \pi_1(Y, y_0) \\ {\scriptstyle \pi_1(f_1)}\downarrow & & \downarrow{\scriptstyle \pi_1(g_2)} \\ \pi_1(X, x_0) & \xrightarrow{\pi_1(g_1)} & \pi_1(Z, z_0) \end{array}$$

is a pushout in Group.

Proof Let us set $\mathring{M}_{f_j} = M_{f_j} \setminus \operatorname{im} i_{f_j}$ for $j = 1, 2$. We have a commutative diagram



whose lower part comes from Theorem 2.22 and the discussion below. All downward pointing arrows are homotopy equivalences. The point $(a_0, \frac{1}{2}) \in A \times (0, 1)$ determines base points in all other spaces. The top face is the pushout in Top of an open cover. Applying the π_1-functor turns it into a pushout in Group by the group version of van Kampen's theorem (Theorem 2.7). Hence also the bottom face becomes a pushout in Group after applying π_1. □

As the easiest application of the theorem, the collapse space of a cofibration $i\colon A \to X$ of path connected nonempty spaces has fundamental group

$$\pi_1(X/A, A/A) \cong \pi_1(X, x_0)/\mathcal{N}(\operatorname{im} \pi_1(i)).$$

For a similar application, let us examine the effect on the fundamental group if one attaches an n-cell to a nonempty space Y as we did in Example 1.40. So let $f\colon (S^{n-1}, \bullet) \to (Y, y_0)$ be a pointed map and consider the pushout

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{f} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ D^n & \longrightarrow & Z. \end{array}$$

By Example 2.18, the inclusion i is a closed cofibration so Theorem 2.27 and Lemma 1.43 prove the following result.

Theorem 2.28

(i) If $n \geq 3$, then $\pi_1(j)\colon \pi_1(Y, y_0) \xrightarrow{\cong} \pi_1(Z, z_0)$ is an isomorphism of groups.
(ii) If $n = 2$, then $\pi_1(j)$ is surjective and $\ker \pi_1(j)$ is the normal subgroup of $\pi_1(Y, y_0)$ generated by the loop $[f] \in \pi_1(Y, y_0)$.

Example 2.29 The (closed connected) **orientable surface of genus g** is defined by the pushout

$$\begin{array}{ccc} S^1 & \xrightarrow{f} & \bigvee_{i=1}^{2g} S^1 \\ \downarrow & & \downarrow \\ D^2 & \longrightarrow & \Sigma_g \end{array}$$

where the attaching map f is determined by the **surface word** $\prod_{i=1}^{g}[a_i, b_i]$ as follows. We subdivide the circle S^1 into $4g$ equal segments and label them (say counterclockwise) by the surface word. We pick orientations of the $2g$ circles in the wedge $\bigvee_{i=1}^{2g} S^1$ and label them by $a_1, b_1, \ldots, a_g, b_g$. Now f is given by mapping segments according to their labels where the exponent ± 1 describes whether the map preserves or reverses orientation. The fundamental group $\pi_1(\Sigma_g, \bullet)$ is called the **surface group of genus *g***. By Theorem 2.28 and Example 2.10, it has the presentation

$$\pi_1(\Sigma_g, \bullet) = \left\langle a_1, b_1, \ldots, a_g, b_g \;\middle|\; \prod_{i=1}^{g}[a_i, b_i] \right\rangle.$$

In particular, we have $\Sigma_0 = S^2$ and $\Sigma_1 \cong \mathbb{T}^2$, hence $\pi_1(\mathbb{T}^2, \bullet) \cong \mathbb{Z}^2$ by Example 1.17. The (closed connected) **nonorientable surface of genus *g*** is defined by the pushout

$$\begin{array}{ccc} S^1 & \xrightarrow{f} & \bigvee_{i=1}^{g} S^1 \\ \downarrow & & \downarrow \\ D^2 & \longrightarrow & N_g \end{array}$$

where f is determined by the surface word $\prod_{i=1}^{g} a_i^2$. So we obtain

$$\pi_1(N_g, \bullet) = \left\langle a_1, \ldots, a_g \;\middle|\; \prod_{i=1}^{g} a_i^2 \right\rangle.$$

In particular, $N_1 = \mathbb{RP}^2$ is the **real projective plane**, which has fundamental group $\pi_1(\mathbb{RP}^2, \bullet) \cong \mathbb{Z}/2\mathbb{Z}$. The nonorientable surface N_2 is also known as the **Klein bottle** to be discussed in Exercise 2.4. The **classification theorem of closed surfaces** asserts that the families Σ_g with $g \geq 0$ and N_g with $g \geq 1$ exhaust all closed connected 2-dimensional manifolds up to homeomorphism.

The method of attaching cells can conversely be used to realize any given group G as the fundamental group of a path connected space. To see this, pick a presentation $G = \langle S|R \rangle$ and consider the pushout

$$\begin{array}{ccc} \coprod_{r \in R} S^1 & \xrightarrow{f} & \bigvee_{s \in S} S^1 \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ \coprod_{r \in R} D^2 & \longrightarrow & X_G. \end{array}$$

Again we have chosen orientations of the circles in $\bigvee_{s \in S} S^1$ and the map f wraps the copy of S^1 corresponding to $r \in R$ along $\bigvee_{s \in S} S^1$ according to the word r. In doing so, we always start and end at a fixed base point $\bullet \in S^1$, which is mapped to the wedge point $\bullet \in \bigvee_{s \in S} S^1$. Let us denote the image of this point in X_G by $x_0 = j(\bullet)$.

Theorem 2.30
We have $\pi_1(X_G, x_0) \cong G$.

Proof The pushout defining X_G factors into two pushouts

$$\begin{array}{ccccc} \coprod_{r\in R} S^1 & \longrightarrow & \bigvee_{r\in R} S^1 & \xrightarrow{f} & \bigvee_{s\in S} S^1 \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle j} \\ \coprod_{r\in R} D^2 & \longrightarrow & \bigvee_{r\in R} D^2 & \longrightarrow & X_G. \end{array}$$

Clearly coproducts in Top of cofibrations are cofibrations. So i is a cofibration by Example 2.18. Hence k is a cofibration by Theorem 2.21 (i). We apply Theorem 2.27 to the right pushout and conclude with Example 2.10 and Lemma 1.43. □

The space X_G is called the **presentation complex** associated with the presentation $\langle S|R\rangle$ of G. We will learn later in Chap. 6 that X_G is an example of a "2-dimensional CW complex." The presentation complex X_G is compact if and only if the presentation $\langle S|R\rangle$ is finite.

2.5 Higher Homotopy Groups

The fundamental group of a pointed space

$$\pi_1(X, x_0) = \{\gamma\colon (I, \{0,1\}) \to (X, x_0)\}/\simeq$$

is defined in terms of pointed homotopy classes of one-dimensional loops and hence encodes primarily low-dimensional data. It is therefore good at distinguishing low-dimensional spaces, for example we have

$$\pi_1(S^1, \bullet) \ncong \pi_1(S^2, \bullet).$$

At the same time, it can have trouble distinguishing high-dimensional spaces as is already visible in the fact $\pi_1(S^n, \bullet) = \{1\}$ for $n \geq 2$. A possible cure is the consideration of **higher homotopy groups** consisting of relative homotopy classes

$$\pi_n(X, x_0) = \{f\colon (I^n, \partial I^n) \to (X, x_0)\}/\simeq$$

where $\partial I^n = I^n \setminus (0,1)^n$ is the boundary of the n-dimensional cube. In the first coordinate, the multiplication in this group is defined by the same formula as in the case of the fundamental group and all other coordinates are left untouched: for $[f], [g] \in \pi_n(X, x_0)$, we set $[f]\cdot[g] = [fg]$ with

$$fg\,(x_1,\ldots,x_n) = \begin{Bmatrix} f(2x_1,x_2,\ldots,x_n) & \text{for } 0 \le x_1 \le \frac{1}{2} \\ g(2x_1-1,x_2,\ldots,x_n) & \text{for } \frac{1}{2} \le x_1 \le 1 \end{Bmatrix}.$$

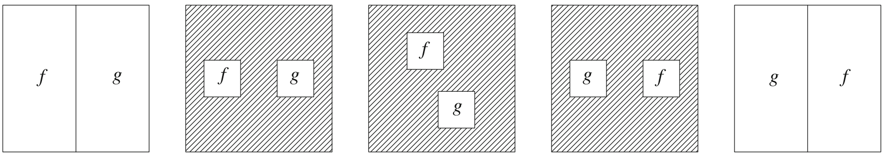
This has the effect that the same proof as for the fundamental group shows that we obtain a well-defined group structure on the set $\pi_n(X, x_0)$ for $n \ge 1$ and that π_n is a functor on $\mathsf{Top}_\bullet$ that factorizes through $\mathsf{HoTop}_\bullet$. For $n = 0$, we formally have $(I^0, \partial I^0) = (\bullet, \emptyset)$, so that $\pi_0(X, x_0) = \pi_0(X)$ is independent of the base point and can be interpreted as the set of path components of X.

For an alternative picture of the group structure on $\pi_n(X, x_0)$, we consider the homeomorphism $I^n/\partial I^n \cong S^n$ that maps the midpoint of I^n to the south pole of S^n and stretches I^n straight upward over the sphere. So the planes $\{x_i = \frac{1}{2}\} \subset I^n$ become mutually orthogonal and equatorially embedded spheres $S^{n-1} \subseteq S^n$ and ∂I^n maps to the north pole. With this identification, a map $(I^n, \partial I^n) \to (X, x_0)$ is the same as a map $(S^n, \bullet) \to (X, x_0)$ and $\pi_n(X, x_0) = \mathrm{Hom}_{\mathsf{HoTop}_\bullet}((S^n, \bullet), (X, x_0))$ as sets. In this visualization, the group multiplication takes the form

$$[f] \cdot [g] = [S^n \to S^n \vee S^n \xrightarrow{f \vee g} S^n]$$

where the first map $S^n \to S^n \vee S^n$ arises from collapsing the equator in S^n corresponding to $\{x_1 = \frac{1}{2}\} \subset I^n$ followed by a homeomorphism from the resulting quotient space to $S^n \vee S^n$, which is defined similarly as above by stretching each of the two hemispheres with collapsed equator over all of S^n.

Higher homotopy groups behave quite differently than the fundamental group. The most striking difference is that for $n \ge 2$, the n-th homotopy group is abelian, so that π_n is in fact a functor $\pi_n \colon \mathsf{HoTop}_\bullet \longrightarrow \mathsf{Ab}$. The following picture describes a relative homotopy from fg to gf in case $n = 2$. The idea carries over to higher dimensions but not to $n = 1$.



The shaded areas and all boundary lines map to the base point x_0. The good news is that higher homotopy groups do distinguish the spheres. We have

$$\pi_k(S^n, \bullet) \cong \begin{Bmatrix} \mathbb{Z}, & k = n, \\ 0, & k < n \end{Bmatrix}$$

as we will see later in Theorem 5.41. Determining the groups $\pi_k(S^n, \bullet)$ for $k > n$ is however a hard problem. In fact, the only simply connected spaces for which all homotopy groups are known are the contractible ones—when all these groups are trivial. The problem is that higher homotopy groups are not well-behaved with respect to homotopy pushouts. There is no immediate generalization of van Kampen's theorem and in fact, the abelian group $\pi_4(S^2 \vee S^2, \bullet)$ is infinite though $\pi_4(S^2, \bullet)$ is not, and the abelian group $\pi_2(S^2 \vee S^1, \bullet)$ is even infinitely generated. So it is desirable to have an invariant capable of distinguishing high-dimensional spaces that at the same time would be practically computable. This is what "homology" accomplishes. In all its variants and disguises, homology shall be the topic of the remainder of the book.

Exercises

2.1 Show that a space X is contractible if and only if the base inclusion $X \subset CX$ embeds X as a strong deformation retract of the cone.

2.2 Let $X = \prod_{\mathbb{R}} I$ be an uncountable product of copies of I, and let $\overline{0} \in X$ be the point that maps to zero under all projections. Show that $\{\overline{0}\} \subset X$ is a closed strong deformation retract but not the zero locus of any map $u: X \to I$.

2.3 Let $x \in S^1$, let $i: S^1 \to S^1 \times S^1$ be the inclusion given by $i(y) = (x, y)$, and let X be the pushout of the diagram $S^1 \times S^1 \xleftarrow{i} S^1 \xrightarrow{i} S^1 \times S^1$.

(a) Find a presentation for $\pi_1(X, \bullet)$ by applying van Kampen's theorem.
(b) Instead of describing X as a pushout of two spaces along a common subspace as above, one can alternatively describe X as the product of two spaces. Find such a description and use it to compute $\pi_1(X, \bullet)$ again. Convince yourself that the two results define isomorphic groups.

2.4 We obtain the **Klein bottle** K from $[0, 1] \times [0, 1]$ by identifying one pair of parallel edges preserving the orientation and the other pair of parallel edges reversing the orientation. Apply van Kampen's theorem to obtain a group presentation $G_1 = \langle a, b \mid R_1 \rangle$ of $\pi_1(K)$ by observing that K can be obtained from gluing a 2-cell to $S^1 \vee S^1$. Explain that the Klein bottle can also be obtained from gluing two Möbius strips along their boundaries. Now obtain a second group presentation $G_2 = \langle c, d \mid R_2 \rangle$ of $\pi_1(K)$ via van Kampen's theorem applied to this description.

2.5 Of course the two group presentations resulting in Problem 2.4 define isomorphic groups. Give an explicit isomorphism and its inverse by writing down the images of a and b as words in c and d and vice versa.

2.6 Let $C \subset \mathbb{R}^2$ be the union of circles of radius $\frac{1}{n}$ with center $(\frac{1}{n}, 0)$ for $n \in \mathbb{N}$. Show that $(C, (0, 0))$ is not a cofibration. Is C homeomorphic to $\bigvee_{n \in \mathbb{N}} S^1$?

3 Homology: Ideas and Axioms

After discussing the challenges and chances of working with higher dimensional generalizations of the fundamental group, we are now ready to give an overall idea of homology as an alternative and more accessible topological tool. We begin with the introduction of **simplicial homology**, a variant of homology that is only defined for a special type of spaces but for which the original, combinatorial idea still shines through. The basic properties of simplicial homology will make apparent that there should be an axiomatic approach to homology theory that we present in the last section.

3.1 The Idea of Homology

The basic building block of homology is the **standard *n*-simplex** defined by

$$\Delta^n = \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} x_i = 1,\ x_i \geq 0 \text{ for all } i = 0, \dots, n \right\}$$

This means the 0-simplex consists of a single point, the 1-simplex is a closed edge, the 2-simplex is a solid equilateral triangle, the 3-simplex is a solid tetrahedron, and so on. If $\{v_0, v_1, \dots, v_n\}$ denotes the standard basis of $\mathbb{R}^{n+1}$, then Δ^n can equivalently be described as the **convex hull** of this basis, denoted by

$$\Delta^n = [v_0, v_1, \dots, v_n]$$

So the basis vectors form the **vertices** of the simplex: v_i is called the ***i*-th vertex**. Dropping the i-th vertex from the convex hull construction gives the ***i*-th face**

$$[v_0, \dots, \widehat{v_i}, \dots, v_n] \subset [v_0, \dots, v_n]$$

H. Kammeyer, *Introduction to Algebraic Topology*, Compact Textbooks in Mathematics, https://doi.org/10.1007/978-3-030-98313-0_3

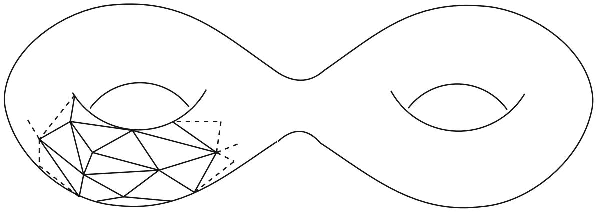


Fig. 3.1 Triangulation of a surface of genus two

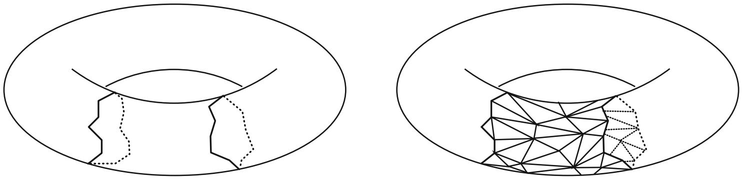


Fig. 3.2 Two homologous 1-cycles in the torus

of Δ^n that opposes v_i, where the "hat" decoration means the corresponding vertex is omitted. Of course, this procedure can be iterated, faces have faces again, until finally 0-simplices have the empty space as the only face by agreement. Various simplices can be glued along faces to form topological spaces known as **simplicial complexes** or, more generally, Δ-**complexes**.

Let X be a space that comes with such a "triangulation" by simplices as in Fig. 3.1.[1] We consider ***n*-chains** of n-simplices in X. For the moment, we think of them as a choice of finitely many n-simplices in X. This would later correspond to working with coefficients in $\mathbb{Z}/2\mathbb{Z}$. Now it is not hard to make explicit the intuitive notion of **boundary** of an n-chain. It is the $(n-1)$-chain consisting of the faces of the n-simplices in the n-chain, which are not the common face of an even number (usually two) of adjacent simplices of the chain. The so obtained boundary might be empty, in which case the n-chain is called an ***n*-cycle**. In particular, the boundary of an $(n+1)$-chain is always an n-cycle. We only have to observe that the boundary of the boundary of any simplex is empty. The n-th homology $H_n(X)$ of X consists of n-cycles up to boundaries of $(n+1)$-chains: We identify two n-cycles if they are **homologous**: if their union forms the boundary of an $(n+1)$-chain. The construction is illustrated in Fig. 3.2.[1]

[1] The image has previously appeared in [16].

3.2 Simplicial Homology

In this section we make the above program precise though we do it right away for integer coefficients. Let $\partial\Delta^n = \bigcup_{i=0}^{n}[v_0, \ldots, \hat{v}_i, \ldots, v_n]$ be the **boundary** of Δ^n and let $(\overset{\circ}{\Delta^n}) = \Delta^n \backslash \partial\Delta^n$ be the **interior** of Δ^n.

Definition 3.1

A **Δ-complex** is a topological space X with a family of maps

$$\{\sigma_\alpha^n : \Delta^n \to X\}$$

for each $n \geq 0$ such that

(i) Each map $\sigma_\alpha^n|_{\overset{\circ}{\Delta^n}}$ is injective.
(ii) Each point $x \in X$ lies in $\mathrm{im}(\sigma_\alpha^n|_{\overset{\circ}{\Delta^n}})$ for exactly one σ_α^n.
(iii) Each restriction $\sigma_\alpha^n|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}$ is equal to some σ_β^{n-1} where we identify $[v_0, \ldots, \hat{v}_i, \ldots, v_n]$ with $\Delta^{n-1} = [v_0, \ldots, v_{n-1}]$ by the unique linear homeomorphism that preserves the order of the vertices.
(iv) A subspace $A \subseteq X$ is open if and only if $(\sigma_\alpha^n)^{-1}(A) \subseteq \Delta^n$ is open for all σ_α^n.

We refer to a map $\sigma_\alpha^n : \Delta^n \to X$ as an **n-simplex** of the Δ-complex X. A Δ-complex X comes with a filtration by **skeleta** $X^0 \subseteq \cdots \subseteq X^k \subseteq \cdots \subseteq X$ where the **k-skeleton** X^k is the union of all images of simplices of dimension at most k. A Δ-complex X is called k-**dimensional** if $X^k = X$ and $X^{k-1} \neq X$ and it is called **infinite dimensional** if there is no such k. We say that X is of **finite type** if for each $k \geq 0$, it has only finitely many k-simplices. We say X is **finite** if it has only finitely many simplices altogether.

A **simplicial complex** is a Δ-complex X in which each simplex is uniquely determined by its vertices. It follows that simplicial complexes have a convenient and entirely combinatorial description as an **abstract simplicial complex**. One starts with a set of vertices and specifies for each finite subset if it spans a simplex or not. In doing so one only has to make sure that for each simplex also all of its faces are simplices. The combinatorial simplicity comes at a price. Even for very simple spaces, one needs a large number of simplices: The 2-torus $\mathbb{T}^2$ as a simplicial complex needs at least 14 triangles, 21 edges, and 7 vertices! To endow $\mathbb{T}^2$ with a Δ-complex structure, two triangles, three edges, and one vertex suffice.

Definition 3.2

Let X be a Δ-complex. The n-th **simplicial chain module** $C_n^\Delta(X)$ is the free $\mathbb{Z}$-module with basis

$$\{\sigma_\alpha^n : \Delta^n \to X\}$$

Accordingly, the n-th chain module consists of formal $\mathbb{Z}$-linear combinations

$$\sum_{\alpha} k_\alpha \sigma_\alpha^n \in C_n^\Delta(X)$$

with finitely many nonzero coefficients. We call them simplicial **n-chains** in X.

Definition 3.3
The **boundary homomorphism** (also known as **differential**)

$$\partial_n : C_n^\Delta(X) \to C_{n-1}^\Delta(X)$$

for $n \geq 1$ is defined on the basis by

$$\partial_n(\sigma_\alpha^n) = \sum_{i=0}^{n} (-1)^i \sigma_\alpha^n|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}.$$

The sign ensures that the boundary of a boundary of an n-chain is trivial as we verify in the following lemma. For $n < 0$, we formally set $C_n^\Delta(X) = 0$ and hence $\partial_n = 0$ for $n \leq 0$.

Lemma 3.4
We have $\partial_{n-1} \circ \partial_n = 0$.

Proof Let $\sigma = \sigma_\alpha^n \in C_n^\Delta(X)$ be any basis element. Then

$$\begin{aligned}\partial_{n-1}(\partial_n(\sigma)) &= \partial_{n-1}\left(\sum_{i=0}^{n} (-1)^i \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}\right) = \\ &= \sum_{j<i} (-1)^i (-1)^j \sigma|_{[v_0,\dots,\hat{v}_j,\dots,\hat{v}_i,\dots,v_n]} + \\ &\quad + \sum_{j>i} (-1)^i (-1)^{j-1} \sigma|_{[v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_n]} = 0\end{aligned}$$

because the two sums have the same summands with opposite signs. □

The lemma shows that $(C_*^\Delta(X), \partial_*)$ is a **chain complex** in the algebraic sense: A sequence of R-modules over a commutative ring R (in our case $R = \mathbb{Z}$)

$$\cdots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots$$

indexed over $n \in \mathbb{Z}$ and connected by homomorphisms ∂_n satisfying the condition $\partial_n \circ \partial_{n+1} = 0$. The latter property can be restated as $\operatorname{im} \partial_{n+1} \subseteq \ker \partial_n$ or in words: "every ***n*-boundary** is an ***n*-cycle**". Homology captures the defect as to whether the converse is true.

Definition 3.5
The **n-th homology** of a chain complex (C_*, ∂_*) is the R-module

$$H_n(C_*) = \ker \partial_n / \operatorname{im} \partial_{n+1}.$$

Definition 3.6
The **n-th simplicial homology** of a Δ-complex X is the $\mathbb{Z}$-module

$$H_n^{\Delta}(X) = H_n(C_*^{\Delta}(X)).$$

Thus two cycles $z_1, z_2 \in Z_n(X) := \ker \partial_n$ represent the same element in $H_n^{\Delta}(X)$ ("are **homologous**") if they differ by a boundary $z_1 - z_2 \in B_n(X) := \operatorname{im} \partial_{n+1}$. Note that a $\mathbb{Z}$-module is the same as an abelian group. More precisely, the categories of abelian groups and the category of $\mathbb{Z}$-modules are isomorphic by the forgetful functor $\mathbb{Z}\text{-mod} \to \mathsf{Ab}$ and the functor $\mathsf{Ab} \to \mathbb{Z}\text{-mod}$, which sends an abelian group $(G, +)$ to the module that is G itself as abelian group and has scalar multiplication defined by $n \cdot g = (g + \cdots + g)$ for $n \in \mathbb{Z}$, which for $n < 0$ means $n \cdot g = -g \cdots - g$. Hence for our choice of coefficients in $\mathbb{Z}$, we can interchangeably talk about **homology modules** or **homology groups**.

Example 3.7 Let us consider the circle $X = S^1$ with the Δ-complex structure given by one 0-simplex v and one 1-simplex e glued to v at both ends. The simplicial chain complex $(C_*^{\Delta}(X), \partial_*)$ looks like

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}e \xrightarrow{\partial_1} \mathbb{Z}v \longrightarrow 0$$

and we have $\partial_1(e) = v - v = 0$. Therefore all differentials in the chain complex are trivial, which has the effect that the homology groups agree with the chain groups

$$H_k^{\Delta}(S^1) \cong \begin{cases} \mathbb{Z} & k = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

Example 3.8 Let $X = \mathbb{RP}^2$ be the real projective plane with the Δ-complex structure pictured schematically in Fig. 3.3. The lower left 0-simplex is the same as the upper right one and we denote it by v. The upper left 0-simplex is the same as the lower right one and we denote it by w. The left and right hand 1-simplices are identical and are denoted by a whereas the upper and lower 1-simplices are likewise identical and are denoted by b. The

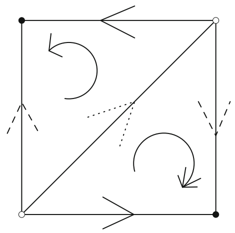


Fig. 3.3 A Δ-complex structure of the real projective plane $\mathbb{RP}^2$. Same arrows describe the same 1-simplex. The curved arrows refer to the orientations of the two 2-simplices that determine the signs of the faces under the boundary map

diagonal 1-simplex is called c. The upper left 2-simplex is called U while the lower right 2-simplex is called L. The arrows point from lower to higher vertex indices. Hence we have

$$\begin{aligned}\partial_2(U) &= -a + b + c\\ \partial_2(L) &= a - b + c\\ \partial_1(a) &= w - v\\ \partial_1(b) &= w - v\\ \partial_1(c) &= v - v = 0\end{aligned}$$

Thus identifying

$$\begin{aligned}C_0^\Delta(X) &= \mathbb{Z}v \oplus \mathbb{Z}w \cong \mathbb{Z}^2\\ C_1^\Delta(X) &= \mathbb{Z}a \oplus \mathbb{Z}b \oplus \mathbb{Z}c \cong \mathbb{Z}^3\\ C_2^\Delta(X) &\cong \mathbb{Z}U \oplus \mathbb{Z}L \cong \mathbb{Z}^2\end{aligned}$$

the simplicial chain complex $C_*^\Delta(X)$ can be described by matrix multiplication as

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\begin{pmatrix}-1 & +1\\ +1 & -1\\ +1 & +1\end{pmatrix}} \mathbb{Z}^3 \xrightarrow{\begin{pmatrix}-1 & -1 & 0\\ +1 & +1 & 0\end{pmatrix}} \mathbb{Z}^2 \longrightarrow 0.$$

We see that the differential ∂_2 is injective, hence $Z_2(X) = \ker \partial_2 = 0$, which gives

$$H_2^\Delta(\mathbb{RP}^2) = 0.$$

For the first differential we compute that $\partial_1 \begin{pmatrix}k\\ l\\ m\end{pmatrix} = 0$ is equivalent to $k + l = 0$ so that

$$\ker \partial_1 = \left\{ \begin{pmatrix} k \\ -k \\ m \end{pmatrix} : k, m \in \mathbb{Z} \right\}.$$

To identify the image of the second differential we compute $\partial_2 \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} -r+s \\ r-s \\ r+s \end{pmatrix}$. Setting $p = -r+s$, therefore $r+s = p+2r$ gives $\operatorname{im} \partial_2 = \left\{ \begin{pmatrix} p \\ -p \\ p+2q \end{pmatrix} : \ p, q \in \mathbb{Z} \right\}$. Substituting $m = k + m'$ above, we see that

$$H_1^{\Delta}(\mathbb{RP}^2) \cong \mathbb{Z}/2\mathbb{Z}.$$

Finally we have

$$\operatorname{im} \partial_1 = \left\{ \begin{pmatrix} k \\ -k \end{pmatrix} : k \in \mathbb{Z} \right\}$$

and thus

$$H_0^{\Delta}(\mathbb{RP}^2) \cong \mathbb{Z}$$

generated by either $\binom{1}{0} + B_0(X)$ or $\binom{0}{1} + B_0(X)$. To sum up, we computed

$$H_k^{\Delta}(\mathbb{RP}^2) \cong \begin{cases} \mathbb{Z} & k = 0 \\ \mathbb{Z}/2\mathbb{Z} & k = 1 \\ 0 & \text{otherwise} \end{cases}$$

For the sake of quick illustration, we set up an ad hoc computation of the homology of the real projective plane. To compute the homology of a chain complex of free finite rank R-modules over a principal ideal domain R algorithmically, one adheres to the following algebraic lemma.

Lemma 3.9
Let $C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$ be a chain complex of free finite rank R-modules over a principal ideal domain R. Then

$$H_1(C_*) \cong R^b \oplus R/(a_1) \oplus \cdots \oplus R/(a_n)$$

where $a_1, \ldots, a_n \in R$ are the ***invariant factors*** *of ∂_2 and where*

$$b = \operatorname{rank} \ker \partial_1 - n.$$

Recall that the invariant factors of a homomorphism of free R-modules over a principal ideal domain R are the up to associatedness uniquely defined nonzero elements $a_1 \mid a_2 \mid \cdots \mid a_n \in R$ such that there exist bases of the domain and codomain with respect to which the homomorphism has a transformation matrix A in **Smith normal form**: $A_{ii} = a_i$ and all other entries zero. There are well-known algorithms that turn a matrix over R into Smith normal form. The reader can find a worked out example computation for a (3×3)-matrix over $R = \mathbb{Z}$ in [7, II.7.8]. Of course $\operatorname{rank}\ker\partial_1$ and $n = \operatorname{rank}\operatorname{im}\partial_2$ can likewise directly be inferred from the Smith normal forms of ∂_1 and ∂_2, respectively.

Proof Since R is a principal ideal domain and since C_1 and C_0 are free of finite rank, the nullspace $C_1^0 = \ker\partial_1$ is complemented. So we obtain a direct sum decomposition $C_1 \cong C_1^0 \oplus C_1^1$ with free summands. Replacing the codomain of ∂_2 by C_1^0, we can find bases of C_2 and C_1^0, which put ∂_2 in Smith normal form. Picking any basis of C_1^1, the Smith normal form of the original ∂_2 is then obtained by extending the matrix with zero rows. Since $H_1(C_*) = C_1^0/\operatorname{im}\partial_2$, the lemma is now clear. □

3.3 Relative Simplicial Homology with Coefficients

It is a common theme in topology, and actually in all of mathematics, to explicitly disregard any information that is not of interest for the problem at hand. For example, one might only want to capture topological properties or phenomena of a Δ-complex X, which are not already produced by a given **subcomplex** $A \subseteq X$. This leads to the notion of **relative simplicial homology** $H_n^\Delta(X, A)$. Additionally, the ring $\mathbb{Z}$ on which we based our construction of simplicial homology so far, might not always be the best choice to work with. For instance, we saw that $H_2^\Delta(\mathbb{RP}^2) \cong 0$, which one might find unfortunate in the sense that our current definition of homology does not seem to recognize the projective plane as a two-dimensional object. The objective of this section therefore is to define relative simplicial homology $H_n^\Delta(X, A; R)$ with coefficients in any commutative ring R such that we recover the previous definition for $A = \emptyset$ and $R = \mathbb{Z}$.

Definition 3.10

Let X be a Δ-complex. A **sub-$\boldsymbol{\Delta}$-complex** (or simply **subcomplex**) is a subspace $A \subseteq X$ given by a union of simplices in X.

Observe that subcomplexes are always closed because $(\sigma_\alpha^n)^{-1}(A)$ is a union of subsimplices of Δ^n and Δ^n has only finitely many subsimplices. We call (X, A) as above a Δ-pair.

Definition 3.11

The **relative simplicial chain module** of a Δ-pair (X, A) is the factor module

$$C_n^\Delta(X, A) = C_n^\Delta(X)/C_n^\Delta(A)$$

Note that chains in A are trivial in $C_n^\Delta(X, A)$ and the differential on $C_n^\Delta(X)$ satisfies $\partial_n(C_n^\Delta(A)) \subseteq C_{n-1}^\Delta(A)$. Thus it descends to a new differential

$$\partial_n : C_n^\Delta(X, A) \longrightarrow C_{n-1}^\Delta(X, A)$$

and we still have $\partial_{n-1} \circ \partial_n = 0$. This shows that the relative simplicial chain modules form a chain complex $(C_*^\Delta(X, A), \partial_*)$. A cycle from $Z_n^\Delta(X, A) := \ker \partial_n$ is now a **relative cycle**, meaning a chain whose boundary lies in A. Two such relative cycles are **relatively homologous** if their difference becomes a boundary in $B_n^\Delta(X, A) := \operatorname{im} \partial_{n+1}$ after adding a chain from A if need be.

Definition 3.12
The **n-th relative simplicial homology** of a Δ-pair (X, A) is the $\mathbb{Z}$-module

$$H_n^\Delta(X, A) = H_n(C_*^\Delta(X, A)).$$

Example 3.13 We consider the case $X = \Delta^n$, $A = \partial\Delta^n$. Then $C_n^\Delta(\Delta^n, \partial\Delta^n) \cong \mathbb{Z}$ whereas $C_k^\Delta(\Delta^n, \partial\Delta^n) \cong 0$ for $k \neq n$, hence

$$H_k^\Delta(\Delta^n, \partial\Delta^n) \cong \begin{cases} \mathbb{Z} & \text{if } k = n, \\ 0 & \text{if } k \neq n \end{cases}$$

More generally, let R be any commutative ring (with unit element $1 \in R$) and let $C_n^\Delta(X; R)$ be the free R-module with basis $\{\sigma_\alpha^n : \Delta^n \to X\}$. Correspondingly, we obtain $C_*^\Delta(X, A; R) = C_*^\Delta(X; R)/C_*^\Delta(A; R)$, called the **relative simplicial chain complex with coefficients in** R. The differential ∂_* is defined as in Definition 3.3, noting that $-1 \in R$ because $1 \in R$. The homology of this chain complex is denoted by $H_n^\Delta(X, A; R)$ and is called **relative simplicial homology with coefficients in** R. Setting $C_n^\Delta(X; R) := C_n^\Delta(X, \emptyset; R)$ and $H_n^\Delta(X; R) := H_n^\Delta(X, \emptyset; R)$, we recover the previous absolute definitions from Sect. 3.2 in the case of $R = \mathbb{Z}$.

Example 3.14 Recall from Example 3.8 that the differential

$$C_2^\Delta(\mathbb{RP}^2; \mathbb{Z}) \xrightarrow{\partial_2} C_1^\Delta(\mathbb{RP}^2; \mathbb{Z})$$

has the form

$$\mathbb{Z}^2 \xrightarrow{\begin{pmatrix} -1 & +1 \\ +1 & -1 \\ +1 & +1 \end{pmatrix}} \mathbb{Z}^3$$

If we replace the ring $\mathbb{Z}$ by $R = \mathbb{Z}/2\mathbb{Z}$, then the equality $-1 = +1$ has the effect that $C_2^\Delta(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\partial_2} C_1^\Delta(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z})$ is of the form

$$(\mathbb{Z}/2\mathbb{Z})^2 \xrightarrow{\begin{pmatrix}1&1\\1&1\\1&1\end{pmatrix}} (\mathbb{Z}/2\mathbb{Z})^3$$

which is a rank-1 linear map of $\mathbb{Z}/2\mathbb{Z}$-vector spaces. By the rank-nullity theorem, also the kernel is one-dimensional whence

$$H_2^\Delta(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}.$$

Intuitively, relative homology $H_n^\Delta(X, A; R)$ quantifies how much $H_n^\Delta(X; R)$ differs from $H_n^\Delta(A; R)$. To make this statement algebraically precise, we introduce some terminology that will turn out to be useful throughout the course.

Definition 3.15

A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of objects and morphisms in R-mod is called **exact** at B if $\ker g = \operatorname{im} f$.

Observe that in this case g is injective if and only if $f = 0$, and f is surjective if and only if $g = 0$. A **long exact sequence** (LES) in R-mod is a sequence

$$\cdots \longrightarrow A_{n+1} \longrightarrow A_n \longrightarrow A_{n-1} \longrightarrow \cdots$$

of objects and morphisms that is everywhere exact.

Theorem 3.16 (LES)
Let (X, A) be a Δ-pair. We have a long exact sequence

$$\cdots \xrightarrow{d_{n+1}} H_n^\Delta(A; R) \xrightarrow{i_n} H_n^\Delta(X; R) \xrightarrow{j_n} H_n^\Delta(X, A; R) \xrightarrow{d_n} H_{n-1}^\Delta(A; R) \xrightarrow{i_{n-1}} \cdots$$

$$\text{where } \begin{cases} i_n \\ j_n \\ d_n \end{cases} \text{is induced by the map } \begin{cases} Z_n^\Delta(A; R) \to Z_n^\Delta(X; R) \\ Z_n^\Delta(X; R) \to Z_n^\Delta(X, A; R) \\ Z_n^\Delta(X, A; R) \xrightarrow{\partial_n} Z_{n-1}^\Delta(A; R) \end{cases}$$

with $Z_n^\Delta(X, A; R) = \ker(\partial_n : C_n^\Delta(X, A; R) \longrightarrow C_{n-1}^\Delta(X, A; R))$ and so forth.

Proof In this condensed proof, we drop the coefficient ring from the notation.

Exactness at $H_n^\Delta(A)$. Let $z + B_n^\Delta(A) \in \ker i_n$. Then $z \in B_n^\Delta(X)$, which means there exists $c \in C_{n+1}^\Delta(X)$ with $\partial_{n+1}c = z$. But $z \in Z_n^\Delta(A) \subseteq C_n^\Delta(A)$, so $c \in Z_n^\Delta(X, A)$, hence $z + B_n^\Delta(A) \in \operatorname{im} d_{n+1}$. Conversely, let $z + B_n^\Delta(A) \in \operatorname{im} d_{n+1}$. Then there exists $c \in C_{n+1}^\Delta(X)$ with $\partial_{n+1}c = z$. Therefore $z \in B_n^\Delta(X)$, hence $z + B_n^\Delta(A) \in \ker i_n$.

Exactness at $H_n^\Delta(X)$. Let $z + B_n^\Delta(X) \in \ker j_n$. Then $z \in B_n^\Delta(X, A)$, which means there exists $c \in C_{n+1}^\Delta(X)$ with $z - \partial_{n+1}c \in C_n^\Delta(A)$ and in fact $z - \partial_{n+1}c \in Z_n^\Delta(A)$

because $\partial_n(z - \partial_{n+1}c) = 0$. We obtain $i_n(z - \partial_{n+1}c + B_n^{\Delta}(A)) = z + B_n^{\Delta}(X)$ and so $z + B_n^{\Delta}(X) \in \operatorname{im} i_n$. Conversely, let $z + B_n^{\Delta}(X) \in \operatorname{im} i_n$. Then there exists $b \in B_n^{\Delta}(X)$ and $z' \in Z_n^{\Delta}(A)$ such that $z' = z + b$. Therefore $z - z' = -b \in B_n^{\Delta}(X)$, hence $z \in B_n^{\Delta}(X, A)$, which shows $z + B_n^{\Delta}(X) \in \ker j_n$.

Exactness at $H_n^{\Delta}(X, A)$. Let $z + B_n^{\Delta}(X, A) \in \ker d_n$. Then $\partial_n z \in B_{n-1}^{\Delta}(A)$. Hence there is $c \in C_n^{\Delta}(A)$ with $\partial_n c = \partial_n z$. So $z - c \in Z_n^{\Delta}(X)$ and $j_n((z-c) + B_n^{\Delta}(X)) = z + B_n^{\Delta}(X, A)$ whence $z + B_n^{\Delta}(X, A) \in \operatorname{im} j_n$. Conversely, let $z + B_n^{\Delta}(X, A) \in \operatorname{im} j_n$. Then there is $z' \in Z_n^{\Delta}(X)$ and $c \in C_n^{\Delta}(A)$ with $z' = z + c$. It follows that $\partial_n(z) = \partial_n(z' - c) = \partial_n(-c) \in B_{n-1}^{\Delta}(A)$ so $z + B_n^{\Delta}(X, A) \in \ker d_n$. □

The long exact homology sequence captures precisely in how far $H_n^{\Delta}(A; R)$, $H_n^{\Delta}(X; R)$, and $H_n^{\Delta}(X, A; R)$ interrelate. We single out an important special case.

Corollary 3.17
The relative homology $H_n^{\Delta}(X, A; R)$ vanishes for all n if and only if all inclusions $H_n^{\Delta}(A) \xrightarrow{i_n} H_n^{\Delta}(X)$ are isomorphisms.

Proof In an exact sequence, only the trivial R-module fits between two adjacent zero morphisms. With the above observation, this gives the "if" part of the corollary. Conversely, the zero morphisms are the only morphism from and to the trivial R-module. Using the observation again, this gives the "only if" part. □

In view of the corollary, one could wonder if we have in general $H_n^{\Delta}(A; R) \subseteq H_n^{\Delta}(X; R)$ and $H_n^{\Delta}(X, A; R) \cong H_n^{\Delta}(X; R)/H_n^{\Delta}(A; R)$. This is however not always true. In fact, it is true if and only if in the long exact sequence

$$\cdots \xrightarrow{d_{n+1}} H_n^{\Delta}(A; R) \xrightarrow{i_n} H_n^{\Delta}(X; R) \xrightarrow{j_n} H_n^{\Delta}(X, A; R) \xrightarrow{d_n} H_{n-1}^{\Delta}(A; R) \xrightarrow{i_n} \cdots$$

the differentials d_{n+1} and d_n are zero. In this case, the three terms of order n form what is called a **short exact sequence** (SES)

$$0 \longrightarrow H_n^{\Delta}(A; R) \xrightarrow{i_n} H_n^{\Delta}(X; R) \xrightarrow{j_n} H_n^{\Delta}(X, A; R) \longrightarrow 0,$$

meaning a five term exact sequence starting and ending in the trivial module. It is then still not necessarily true that i_n and j_n can be identified with the canonical inclusion and projection of a direct sum decomposition

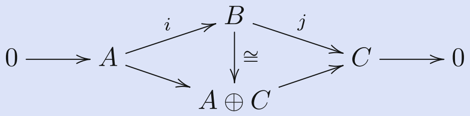
$$H_n^{\Delta}(X; R) \cong H_n^{\Delta}(A; R) \oplus H_n^{\Delta}(X, A; R).$$

In fact, this holds true if and only if the SES **splits**.

Lemma 3.18 (Splitting Lemma)

Let $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$ *be a short exact sequence of* R*-modules. The following are equivalent:*

(i) There exists a homomorphism $p\colon B \longrightarrow A$ *such that* $p \circ i = \mathrm{id}_A$.
(ii) There exists a homomorphism $s\colon C \longrightarrow B$ *such that* $j \circ s = \mathrm{id}_C$.
(iii) There exists an isomorphism $B \cong A \oplus C$ *fitting into the diagram*



where the lower arrows are the canonical inclusion and projection.

Proof Statement (iii) is formally the strongest. It clearly implies (i) and (ii). To see that (ii) implies (iii), we note that i is injective while j vanishes on $\operatorname{im} i$ and restricts to an isomorphism on $\operatorname{im} s$. So it is enough to show that B is the internal direct sum of $\operatorname{im} i$ and $\operatorname{im} s$. To see that, we decompose $b \in B$ as $b = (b - s(j(b))) + s(j(b))$ with $b - s(j(b)) \in \ker j = \operatorname{im} i$. If $b \in \operatorname{im} i \cap \operatorname{im} s$, then by surjectivity of j there exists $b' \in B$ with $b = s(j(b'))$. Since $\operatorname{im} i = \ker j$, we have $0 = j(b) = j(s(j(b'))) = j(b')$ so $b = s(0) = 0$. Similarly, to see that (i) implies (iii), we show that B is the internal direct sum of $\operatorname{im} i$ and $\ker p$. For $b \in B$, we decompose $b = i(p(b)) + (b - i(p(b)))$ with $b - i(p(b)) \in \ker p$. If $b \in \operatorname{im} i \cap \ker p$, then there is $a \in A$ such that $b = i(a)$ and $0 = p(b) = p(i(a)) = a$, so $b = i(0) = 0$. Since $\operatorname{im} i = \ker j$, it follows that j restricts to an isomorphism on $\ker p$ whence (iii) follows. □

The SES is called **split** if it satisfies one (then all) of the above conditions. By the characterization in (ii), a SES always splits if C is a free R-module. The splitting lemma can be stated and proven in a purely categorical manner for so called **abelian categories**. In contrast, the splitting lemma fails in Group because non-abelian groups can decompose as **semidirect products**. For example, the sign of permutations gives a SES

$$1 \longrightarrow A_3 \longrightarrow S_3 \xrightarrow{\mathrm{sgn}} \{\pm 1\} \longrightarrow 1$$

for the symmetric group on three letters S_3. This SES satisfies (ii) but not (i) because $\{\pm 1\}$ does not lift to a normal subgroup of S_3.

Even in those cases when the LES for relative homology does not decompose into split SESes, it is still a valuable tool to reduce the computation of homology groups to previously obtained results. The second powerful method to compute relative homology uses that homology remains unaffected if one **excises** a subcomplex.

Theorem 3.19 (Excision)
Let X be a Δ-complex and let $A \subseteq Y \subseteq X$ be subcomplexes, such that $A \subseteq \mathring{Y}$. Then the inclusion of Δ-pairs

$$(X \backslash \mathring{A}, Y \backslash \mathring{A}) \xrightarrow{j} (X, Y)$$

induces isomorphisms

$$H_n^\Delta(X \backslash \mathring{A}, Y \backslash \mathring{A}; R) \xrightarrow{\cong} H_n^\Delta(X, Y; R)$$

for all n, where the "$\circ$"-decoration indicates the interior as subspace of X.

Proof We show the stronger statement that the inclusion induces an isomorphism

$$C_*^\Delta(X \backslash \mathring{A}, Y \backslash \mathring{A}; R) \xrightarrow{j_*} C_*^\Delta(X, Y; R)$$

of chain complexes. To see that j_* is surjective, we can write $c + C_n^\Delta(Y; R) \in C_n^\Delta(X, Y; R)$ as $c' + C_n^\Delta(Y, R)$ where c' has no nonzero coefficients for simplices in Y. Since $X \backslash Y \subseteq X \backslash \mathring{A}$, the element $c' + C_n^\Delta(Y \backslash \mathring{A}; R)$ is a preimage of $c + C_n^\Delta(Y; R)$ in $C_n^\Delta(X \backslash \mathring{A}, Y \backslash \mathring{A}; R)$ under j_*. To see injectivity, let $j_*(c + C_n^\Delta(Y \backslash \mathring{A}; R)) = 0$. Then $c \in C_n^\Delta((X \backslash \mathring{A}) \cap Y; R)$. Since $A \subseteq \mathring{Y}$, we have $(X \backslash \mathring{A}) \cap Y \subseteq Y \backslash \mathring{A}$ thus $c \in C_n^\Delta(Y \backslash \mathring{A}; R)$, which shows that $c + C_n^\Delta(Y \backslash \mathring{A}; R)$ is trivial in $C_n^\Delta(X \backslash \mathring{A}, Y \backslash \mathring{A}; R)$. Since j_* is induced by the inclusion j of Δ-pairs, it is clear that j_* commutes with the boundary homomorphisms in $C_*^\Delta(X \backslash \mathring{A}, Y \backslash \mathring{A}; R)$ and $C_*^\Delta(X, Y; R)$. □

3.4 The Eilenberg–Steenrod Axioms for Homology

The most obvious disadvantage of simplicial homology is that unlike homotopy groups, it is only defined for Δ-complexes, not for general topological spaces. One might not feel too discouraged about this as examples of Δ-complexes abound. In particular, all smooth manifolds admit a Δ-structure as one can see with quite some effort. However, Δ-complexes are difficult to handle as category. Even defining morphisms in such a way that $H_n^\Delta(-, -; R)$ becomes a functor $\Delta\text{-pairs} \to R\text{-mod}$ is troublesome though it is much easier for simplicial complexes as we shall see in the next section. It would then moreover not be clear whether homotopic morphisms $f \simeq g$ satisfy $H_n^\Delta(f; R) = H_n^\Delta(g; R)$. This however is a desirable feature as it would allow the conclusion that homotopy equivalent Δ-complexes have isomorphic homology. So ideally, we want homology to be a family of functors

$$H_n : \mathsf{Top}^{(2)} \longrightarrow R\text{-mod}$$

indexed over integers $n \in \mathbb{Z}$ such that we have

(1) A factorization

$$\begin{array}{ccccc} & & \mathsf{HoTop}^{(2)} & & \\ & \nearrow & & \searrow & \\ \mathsf{Top}^{(2)} & & \xrightarrow{\quad H_n \quad} & & R\text{-}\mathsf{mod}, \end{array}$$

(2) A LES for every pair (X, A).
(3) An excision isomorphism for triples (X, Y, A) with $\overline{A} \subseteq \mathring{Y}$.

Here it is apparent how to define the category $\mathsf{HoTop}^{(2)}$. The **Eilenberg–Steenrod** axioms for homology make the above wish list precise.

Definition 3.20
A **homology theory** with values in R-mod consists of a family $(H_n)_{n\in\mathbb{Z}}$ of functors

$$H_n : \mathsf{Top}^{(2)} \longrightarrow R\text{-}\mathsf{mod}$$

and a family of natural transformations

$$\partial_n : H_n \longrightarrow H_{n-1} \circ J,$$

where $J : \mathsf{Top}^{(2)} \to \mathsf{Top}^{(2)}$ is the functor that sends a pair (X, A) to $(A, \emptyset)$ and restricts a map of pairs $f : (X, A) \to (Y, B)$ to $f|_A : (A, \emptyset) \to (B, \emptyset)$. The functors and the natural transformations are required to satisfy the following set of axioms:

(1) **Homotopy invariance**: If two maps of pairs $f, g : (X, A) \to (Y, B)$ are homotopic by a homotopy h with $h_t(A) \subseteq B$ for all $t \in I$, then $H_n(f) = H_n(g)$.
(2) **Long exact sequence**: Let the pair (X, A) determine the inclusions $i : (A, \emptyset) \to (X, \emptyset)$ and $j : (X, \emptyset) \to (X, A)$. Then the sequence

$$\cdots \to H_n(A, \emptyset) \xrightarrow{H_n(i)} H_n(X, \emptyset) \xrightarrow{H_n(j)} H_n(X, A) \xrightarrow{\partial_n(X,A)} H_{n-1}(A, \emptyset) \to \cdots$$

is exact.
(3) **Excision**: Let $A, Y \subseteq X$ be subspaces with $\bar{A} \subseteq \mathring{Y}$. Then the inclusion

$$(X \backslash A, Y \backslash A) \xrightarrow{j} (X, Y)$$

induces isomorphisms

$$H_n(X \backslash A, Y \backslash A) \xrightarrow{H_n(j)} H_n(X, Y)$$

for each $n \in \mathbb{Z}$.

We say that a homology theory satisfies the **dimension axiom** if in addition

(4) $H_n(\bullet, \emptyset) = 0$ for $n \neq 0$.

We will write $H_n(X)$ for $H_n(X, \emptyset)$. The R-modules $H_n(\bullet)$ are called the **coefficient modules** of the homology theory. A homology theory is called **ordinary** if it satisfies the dimension axiom and **generalized** otherwise. Naturality of ∂_n says explicitly that for every map of pairs $f\colon (X, A) \to (Y, B)$, the diagram

$$\begin{CD} H_n(X,A) @>{\partial_n}>> H_{n-1}(A) \\ @V{H_n(f)}VV @VV{H_{n-1}(f|_A)}V \\ H_n(Y,B) @>>{\partial_n}> H_{n-1}(B) \end{CD}$$

commutes. It follows that $H_*(f)$ maps the LES of (X, A) to the LES of (Y, B) so that we obtain a commutative horizontal ladder with exact rows

$$\begin{CD} \cdots @>>> H_n(A) @>>> H_n(X) @>>> H_n(X,A) @>>> H_{n-1}(A) @>>> \cdots \\ @. @V{H_n(f|_A)}VV @V{H_n(f)}VV @V{H_n(f)}VV @V{H_{n-1}(f|_A)}VV \\ \cdots @>>> H_n(B) @>>> H_n(Y) @>>> H_n(Y,B) @>>> H_{n-1}(Y) @>>> \cdots \end{CD}$$

The leftmost square commutes because the underlying square in Top commutes

$$\begin{CD} (A,\emptyset) @>{i}>> (X,\emptyset) \\ @V{f|_A}VV @VV{f}V \\ (B,\emptyset) @>>{i}> (Y,\emptyset) \end{CD}$$

and similarly for the middle square.

The goal suggests itself to construct an ordinary homology theory with values in R-mod and we will do so in the next chapter. In the chapter after that, we verify that this homology theory is isomorphic to simplicial homology for Δ-complexes and naturally so for simplicial complexes.

3.5 Simplicial Approximation

To make the last naturality statement meaningful, we need to discuss in what sense also simplicial homology is functorial. First we describe the morphisms in the category of simplicial complexes.

Definition 3.21

A map $f\colon X \to Y$ of simplicial complexes X and Y is called **simplicial** if it restricts to a map of the 0-skeleta $f^0\colon X^0 \to Y^0$ such that

(i) Whenever $v_0, \dots, v_n \in X^0$ span a simplex in X, so do $f(v_0), \dots, f(v_n)$ in Y.
(ii) The restriction of f to a simplex $[v_0, \dots, v_n]$ is the unique affine linear extension of f^0 so that $f(\sum_{i=0}^n t_i v_i) = \sum_{i=0}^n t_i f(v_i)$.

The two conditions ensure that a simplicial map f is determined by its restriction f^0 to the vertices so that not only simplicial complexes but also simplicial maps have an entirely combinatorial description. A simplicial map $f\colon X \to Y$ induces a homomorphism $C_n^\Delta(f)\colon C_n^\Delta(X) \to C_n^\Delta(Y)$ as follows. Given an n-simplex $\sigma_\alpha^n = [v_0, \dots, v_n]$ in X, we set $C_n^\Delta(f)(\sigma_\alpha^n) = 0$ whenever the vertices $f(v_0), \dots, f(v_n)$ span a simplex of dimension less than n. Otherwise, let $\sigma_\beta^n = [w_0, \dots, w_n]$ be the n-simplex in Y spanned by $f(v_0), \dots, f(v_n)$. We set $C_n^\Delta(f)(\sigma_\alpha^n) = +\sigma_\beta^n$ if the $(n+1)$-tuples $(w_0, \dots, w_n)$ and $(f(v_0), \dots, f(v_n))$ differ by an even permutation and $C_n^\Delta(f)(\sigma_\alpha^n) = -\sigma_\beta^n$ if they differ by an odd permutation. Note that here we use that our definition of simplicial complex gives a little more than just an abstract simplicial complex. The maps $\sigma_\alpha^n\colon \Delta^n \to X$ induce an ordering of the vertices and hence an **orientation** of the simplex defined as an ordering of the vertices up to even permutations. We check that the family of homomorphisms $C_*^\Delta(f)\colon C_*^\Delta(X; R) \to C_*^\Delta(Y; R)$ is a **chain map** of chain complexes, meaning the diagram

$$\begin{array}{ccc} C_n^\Delta(X;R) & \xrightarrow{\partial} & C_{n-1}^\Delta(X;R) \\ \downarrow{\scriptstyle C_n^\Delta(f)} & & \downarrow{\scriptstyle C_{n-1}^\Delta(f)} \\ C_n^\Delta(Y;R) & \xrightarrow{\partial} & C_{n-1}^\Delta(Y;R) \end{array}$$

commutes for every n. The only critical case to consider occurs when f maps an n-simplex $\sigma_\alpha^n = [v_0, \dots, v_n]$ to an $(n-1)$-simplex in Y. Then f maps precisely two of the vertices $v_0, \dots, v_n$ to the same vertex in Y and so $C_{n-1}^\Delta(f)(\partial\sigma_\alpha^n)$ consists of precisely two summands that differ by sign, either because of the definition of ∂ or because of the definition of $C_{n-1}^\Delta(f)$ depending on whether the indices of the two vertices are an odd or an even number apart. So both compositions in the diagram are zero on σ_α^n as required. For a simplicial map of simplicial pairs $f\colon (X, A) \to (Y, B)$ with subcomplexes A and B such that $f(A) \subseteq B$, the map $C_n^\Delta(f)$ descends to a chain map $C_n^\Delta(X, A; R) \to C_n^\Delta(Y, B; R)$ of relative chain complexes. By definition, a chain map sends cycles to cycles and boundaries to boundaries. Hence a simplicial map $f\colon (X, A) \to (Y, B)$ induces a homomorphism $H_n^\Delta(f)\colon H_n^\Delta(X, A; R) \to H_n^\Delta(Y, B; R)$ in simplicial homology. The functor relations $H_n^\Delta(g \circ f) = H_n^\Delta(g) \circ H_n^\Delta(f)$ and $H_n^\Delta(\mathrm{id}_{(X,A)}) = \mathrm{id}_{H_n^\Delta(X,A;R)}$ are readily verified.

The condition for a map $f\colon X \to Y$ to be simplicial is topologically quite restrictive. It is therefore reassuring to know that any continuous map $f\colon X \to Y$

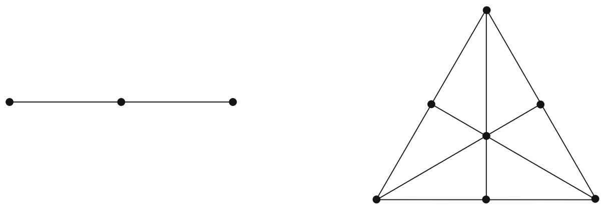


Fig. 3.4 Barycentric subdivision of a 1- and a 2-simplex

from a finite simplicial complex to any simplicial complex is homotopic to a simplicial map on some repeated **barycentric subdivision** of X. To explain this, let $v_b = \frac{1}{n+1}\sum_{i=0}^{n} v_i$ be the **barycenter** (center of mass) of the standard n-simplex $\Delta^n = [v_0, \dots, v_n]$. The **barycentric subdivision** of Δ^n consists of the simplices $[v_b, w_1, \dots, w_n]$ where $[w_1, \dots, w_n]$ runs through the $(n-1)$-simplices in the barycentric subdivision of the $n+1$ faces $[v_0, \dots, \widehat{v_i}, \dots, v_n]$ of Δ^n. This inductive definition ends with the agreement that the barycentric subdivision of $[v_0]$ shall be just $[v_0]$ itself. A visualization for $n = 1$ and $n = 2$ is given in Fig. 3.4.

The **barycentric subdivision** $X^{[1]}$ of a Δ-complex X has the same underlying topological space as X, but each n-simplex $\sigma_\alpha^n \colon \Delta^n \to X$ is replaced by the $(n+1)!$ restrictions to the simplices of the barycentric subdivision of Δ^n. Here again, we identify the simplex $[v_b, w_1, \dots, w_n]$ in the subdivision with the standard simplex $[v_0, \dots, v_n]$ by the unique linear homeomorphism that preserves the order of the vertices. Inductively, we set $X^{[r]} = (X^{[r-1]})^{[1]}$ and $X^{[0]} = X$.

Theorem 3.22 (Simplicial Approximation)
Let X and Y be simplicial complexes and assume that X is finite. Then for every continuous map $f \colon X \to Y$, there exists $r \geq 0$ and a simplicial map $g \colon X^{[r]} \to Y$ such that $f \simeq g$.

It is clear that the iterated barycentric subdivision in the theorem is necessary. For there are only finitely many simplicial maps from the triangle to itself, but there are infinitely many homotopy classes of maps $S^1 \to S^1$ since $\pi_1(S_1, \bullet) \cong \mathbb{Z}$. The proof of the simplicial approximation theorem needs some preparation. To begin, let us endow the finite simplicial complex X with any metric d in which each n-simplex of X is isometric to some embedded n-simplex in $\mathbb{R}^{n+1}$. A possible such choice comes from realizing X as subcomplex of $\Delta^{|X^0|} \subset \mathbb{R}^{|X^0|+1}$.

Lemma 3.23
Let σ be an n-simplex in X and let τ be an n-simplex from the subdivision of σ in $X^{[1]}$. Then $\operatorname{diam}(\tau) \leq \frac{n}{n+1} \cdot \operatorname{diam}(\sigma)$.

Proof It is apparent that the diameter of a simplex is the maximal distance of any two of its vertices. If neither of two vertices of τ is the barycenter of σ, then these two points lie in one of the faces of σ, so we are done by induction. The distance from the barycenter b of σ to any of the vertices v occurring in τ is maximal if v is a vertex of $\sigma = [v_0, \dots, v_n]$, so say $v = v_i$. The line through v_i and b intersects the i-th face $[v_0, \dots, \widehat{v_i}, \dots, v_n]$ of σ in its barycenter $b_i = \frac{1}{n}(v_0 + \dots + \widehat{v_i} + \dots + v_n)$ and hence $b = \frac{n}{n+1}b_i + \frac{1}{n+1}v_i$. So the line segment from v_i to b forms the $\frac{n}{n+1}$-th part of the line segment from v_i to b_i. This shows $\|b - v_i\| = \frac{n}{n+1}\|b_i - v_i\| \leq \frac{n}{n+1}\operatorname{diam}(\sigma)$. □

For a point $x \in X$, we define the **carrier** $\operatorname{carr}(x)$ as the unique simplex in X of which x is an interior point. We say that a simplicial map $g\colon X \to Y$ is a **simplicial approximation** to the continuous map $f\colon X \to Y$ if $g(x) \in \operatorname{carr}(f(x))$ for all $x \in X$. The condition makes it possible to define a homotopy by the formula $H(x, t) = tg(x) + (1 - t)f(x)$, which shows $f \simeq g$. We define the **open star** $\operatorname{st}(v)$ of a vertex $v \in X^0$ as the subset of X consisting of all $x \in X$ whose carrier contains v. To see that $\operatorname{st}(v) \subseteq X$ is indeed open, it is enough to observe that the complement is a union of simplices, hence closed by Definition 3.1 (iv). So we show that $x \notin \operatorname{st}(v)$ implies that $\operatorname{carr}(x)$ is disjoint from $\operatorname{st}(v)$. Indeed, the carrier of a given point is the smallest simplex containing it, so for $y \in \operatorname{carr}(x)$, we have $\operatorname{carr}(y) \subseteq \operatorname{carr}(x)$. Since $v \notin \operatorname{carr}(x)$, this implies $v \notin \operatorname{carr}(y)$ whence $y \notin \operatorname{st}(v)$ which is what we wanted to show. Finally, we observe that an intersection of open stars $\bigcap_{i=0}^{n} \operatorname{st}(v_i)$ is nonempty if and only if $[v_0, \dots, v_n]$ is a simplex in X. For the carrier of a point $x \in \bigcap_{i=0}^{n} \operatorname{st}(v_i)$ contains each vertex $v_0, \dots, v_n$, hence $[v_0, \dots, v_n]$ is a subsimplex of $\operatorname{carr}(x)$. Conversely, each interior point of $[v_0, \dots, v_n]$ is contained in $\bigcap_{i=0}^{n} \operatorname{st}(v_i)$.

Proof of Theorem 3.22 By the above, we are left with the task of constructing a simplicial approximation g of f on some repeated barycentric subdivision $X^{[r]}$. Since a finite simplicial complex is apparently compact, we can pick a Lebesgue-δ for the open cover $\{f^{-1}(\operatorname{st}(w_i))\colon w_i \in Y^0\}$ of X. By Lemma 3.23, there is $r \geq 0$ such that all simplices in $X^{[r]}$ have diameter less than $\frac{\delta}{2}$. For every fixed vertex $v \in (X^{[r]})^0$ and every $x \in \operatorname{st}(v)$, we have $v \in \operatorname{carr}(x)$, hence $d(x, v) < \frac{\delta}{2}$. By the triangle inequality, this gives $\operatorname{diam}(\operatorname{st}(v)) < \delta$. This shows that there is $w_i \in Y^0$ such that $f(\operatorname{st}(v)) \subseteq \operatorname{st}(w_i)$. So we set $g(v) = w_i$ and verify that this choice defines a simplicial map. Let $[v_0, \dots, v_n]$ be a simplex in $X^{[r]}$. Then $\bigcap_{i=0}^{n} \operatorname{st}(v_i)$ is nonempty. Since

$$f\left(\bigcap_{i=0}^{n} \operatorname{st}(v_i)\right) \subseteq \bigcap_{i=0}^{n} f(\operatorname{st}(v_i)) \subseteq \bigcap_{i=0}^{n} \operatorname{st}(g(v_i)),$$

it follows that $\bigcap_{i=0}^{n} \operatorname{st}(g(v_i))$ is nonempty, too, hence $[g(v_0), \dots, g(v_n)]$ is a simplex in Y, as required. Write a given element $x \in X^{[r]}$ with carrier $\operatorname{carr}(x) = [v_0, \dots, v_n]$ in barycentric coordinates $x = \sum_{i=0}^{n} t_i v_i$ with $t_0 + \dots + t_n = 1$. For each $i = 0, \dots, n$, we have $v_i \in \operatorname{carr}(x)$ so $x \in \operatorname{st}(v_i)$, thus $f(x) \in \operatorname{st}(g(v_i))$, which means $g(v_i) \in \operatorname{carr}(f(x))$. This shows $g(x) = \sum_{i=0}^{n} t_i g(v_i) \in \operatorname{carr}(f(x))$ as well. So $g\colon X^{[r]} \to Y$ is a simplicial approximation of f. □

Exercises

3.1 Show that the minimal simplicial structure on the n-sphere S^n has $\binom{n+2}{k+1}$ k-simplices for $k = 0, \dots, n$.

3.2 Find Δ-complex structures for the torus $\mathbb{T}^2$ and the Klein bottle K and compute the simplicial homology.

3.3 Let G be a group with presentation $G = \langle S|R \rangle$ and let X_G be the associated presentation complex. Endow X_G with the structure of a Δ-complex and show that $H_1^{\Delta}(X_G)$ is isomorphic to the abelianization of G.

3.4 Prove or disprove the following assertions.

(a) If the sequence of $\mathbb{Z}$-modules $0 \to M \longrightarrow N \longrightarrow \mathbb{Z} \to 0$ is exact, then $N \cong M \oplus \mathbb{Z}$.
(b) If the sequence of $\mathbb{Z}$-modules $0 \to M \longrightarrow N \longrightarrow \mathbb{Z}/2 \to 0$ is exact, then $N \cong M \oplus \mathbb{Z}/2$.
(c) If the sequence of $\mathbb{Z}/2$-modules $0 \to M \longrightarrow N \longrightarrow \mathbb{Z}/2 \to 0$ is exact, then $N \cong M \oplus \mathbb{Z}/2$.

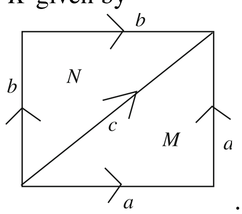
3.5 Let R be a principal ideal domain (e.g., $R = \mathbb{Z}$) and let C be an R-module. Show that if every SES $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$ splits, then C is free.

3.6 Let R be a commutative ring and let M be an R-module.

(a) Show that the functor $(-) \otimes_R M$ is **right exact**: It preserves exactness of $A \to B \to C \to 0$.
(b) Show that the functor $\mathrm{Hom}_R(M, -)$ is **left exact**: It preserves exactness of $0 \to A \to B \to C$.

Hint: Review Exercises 1.3 and 1.7(c).

3.7 Consider the Δ-complex K given by



(a) Explain that K is the Klein bottle and that the subcomplex given by the 2-simplex M and its subsimplices is an embedded Möbius strip.

(b) Compute the absolute and relative simplicial homology $H_*^\Delta(M; R)$, $H_*^\Delta(K; R)$ and $H_*^\Delta(K, M; R)$ for $R = \mathbb{Z}$ and $R = \mathbb{Z}/2\mathbb{Z}$.
(c) Your results make the objects in the long exact sequence of the Δ-pair (K, M) explicit. Find all the homomorphisms for both coefficient rings.

3.8 Let (H_*, ∂_*) be a homology theory with values in R-mod. Let (X, A) be a pair of spaces and suppose that X retracts onto A. Show that the component at (X, A) of the natural transformation ∂_n is trivial for all n.

3.9 Let X be a Δ-complex. Show that $X^{[2]}$ is a simplicial complex.

Singular Homology

4

In this chapter, we construct **singular homology**, an ordinary homology theory with values in R-mod in the sense of Eilenberg–Steenrod. To do so, we would like to adhere to our previous idea of constructing homology by considering *chains* of simplices and define homology as *cycles* up to *boundaries*. To make this program work for a general topological space X, it seems that the only chance to come up with a functorial construction—involving no choices—is to audaciously consider all possible "simplices" at once. In other words, the n-th chain module $C_n(X)$ should be the free R-module with basis the set of all continuous maps $\Delta^n \to X$. Of course, continuous maps can be alarmingly irregular, just think of space filling curves. Moreover, we do not require that $\Delta^n \to X$ would be injective, so the image can be pinched and distorted in various ways. This explains the word *singular*: every continuous map $\Delta^n \to X$ defines a simplex, as singular as it may be. For a typical space X, like a manifold, the chain module $C_n(X)$ will be uncountably generated and the same goes for its submodule of cycles $Z_n(X)$. However, also the submodule of boundaries $B_n(X)$ will be humongous. So one may still hope that the quotient $H_n(X) = Z_n(X)/B_n(X)$ has something useful to say.

4.1 The Definition of Singular Homology

As we just alluded, the construction of singular homology, like simplicial homology, is a two step procedure of defining functors

$$\mathsf{Top}^{(2)} \xrightarrow{C_*(-;R)} R\text{-chain} \xrightarrow{H_*} R\text{-mod}$$

Here a morphism $f_*\colon (C_*, c_*) \to (D_*, d_*)$ in the category R-chain of chain complexes of R-modules is a **chain map**, a family of homomorphisms $f_n\colon C_n \to D_n$ such that $d_n \circ f_n = f_{n-1} \circ c_n$ for all $n \in \mathbb{Z}$. So a chain map gives rise to a commutative ladder

H. Kammeyer, *Introduction to Algebraic Topology*, Compact Textbooks in Mathematics, https://doi.org/10.1007/978-3-030-98313-0_4

$$\begin{array}{ccccccccc}\cdots & \longrightarrow & C_{n+1} & \xrightarrow{c_{n+1}} & C_n & \xrightarrow{c_n} & C_{n-1} & \xrightarrow{c_{n-1}} & \cdots \\ & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\ \cdots & \longrightarrow & D_{n+1} & \xrightarrow{d_{n+1}} & D_n & \xrightarrow{d_n} & D_{n-1} & \xrightarrow{d_{n-1}} & \cdots\end{array}$$

To construct the second functor, we observe again that f_n maps cycles in C_n to cycles in D_n and boundaries in C_n to boundaries in D_n, so a chain map f_* induces morphisms

$$H_n(f_*)\colon H_n(C_*) \to H_n(D_*)$$

and we clearly have $H_n(f_* \circ g_*) = H_n(f_*) \circ H_n(g_*)$ and $H_n(\mathrm{id}_{C_*}) = \mathrm{id}_{H_n(C_*)}$. To construct the first functor, let X be any topological space.

Definition 4.1
The **n-th singular chain module** $C_n(X; R)$ is the free R-module with basis $\{\sigma^n : \Delta^n \to X,\ \sigma^n \text{ continuous}\}$.

Sometimes we will add a superscript and write $C_n^{\mathrm{sing}}(X; R)$ to stress we are dealing with the singular chain module if otherwise confusion is likely. Let us moreover agree that $C_n(X; R) = 0$ for $n < 0$. The same formula as for simplicial homology defines the **singular boundary homomorphism** or **singular differential**

$$\partial_n\colon C_n(X; R) \longrightarrow C_{n-1}(X; R), \quad \partial_n(\sigma^n) = \sum_{i=0}^{n} (-1)^i \sigma^n|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}.$$

It is again understood that we identify $[v_0, \ldots, \hat{v}_i, \ldots, v_n]$ with Δ^{n-1} by the unique linear homeomorphism that preserves the order of the vertices. The same calculation as before gives $\partial_{n-1} \circ \partial_n = 0$. The transition from the absolute to the relative case is now similar to what we did for simplicial homology so that we walk through the process at a swift pace.

- For a pair of spaces (X, A), we define $C_n(X, A; R) = C_n(X; R)/C_n(A; R)$.
- A map of spaces $f\colon (X, A) \to (Y, B)$ induces a homomorphism

$$C_n(f; R)\colon C_n(X, A; R) \to C_n(Y, B; R),\ [\sigma^n : \Delta^n \to X] \mapsto [f \circ \sigma^n]$$

  because $f \circ \sigma^n$ maps to B if σ^n maps to A.
- By construction, $C_n(-; R)\colon \mathsf{Top}^{(2)} \to R\text{-}\mathsf{mod}$ is a functor.
- The differential ∂_n descends to $\partial_n : C_n(X, A; R) \longrightarrow C_{n-1}(X, A; R)$.
- Clearly $\partial_n \circ C_n(f; R) = C_{n-1}(f; R) \circ \partial_n$, so $C_*(f; R)$ is a chain map.

The last point verifies that $C_*(-; R)\colon \mathsf{Top}^{(2)} \longrightarrow R\text{-}\mathsf{chain}$ is a functor. So we are now in the position to make our proposed definition official.

Definition 4.2
The **n-th singular homology** with coefficients in R of a pair of spaces (X, A) is the R-module

$$H_n^{\text{sing}}(X, A; R) = H_n(C_*(X, A; R)).$$

Being a composition of functors, $H_n^{\text{sing}}\colon \mathsf{Top}^{(2)} \to R\text{-}\mathsf{mod}$ is itself a functor. To show that it defines a homology theory, we still have to produce the components

$$\partial_n(X, A)\colon H_n^{\text{sing}}(X, A; R) \longrightarrow H_{n-1}^{\text{sing}}(A; R)$$

of a natural transformation. It turns out that this is merely a task of algebra, and the construction will give the long exact sequence of a pair at one fell swoop.

4.2 The Long Exact Sequence of a Pair of Spaces

The beginning of the field of **homological algebra** is the following lemma that also has applications outside topology. The somewhat sophisticated appearance has made it feature prominently in the opening scene of the 1980 American movie "It's my turn" starring Jill Clayburgh and Michael Douglas.

Lemma 4.3 (Snake Lemma)
A diagram in R-mod

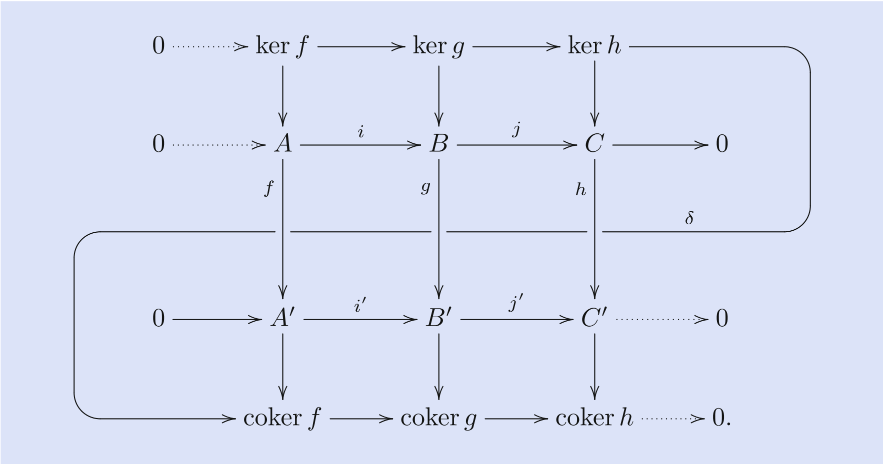
$$\begin{array}{ccccccccc} 0 & \dashrightarrow & A & \xrightarrow{i} & B & \xrightarrow{j} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \dashrightarrow & 0 \end{array}$$

with exact rows induces a natural exact sequence

$$0 \dashrightarrow \ker f \longrightarrow \ker g \longrightarrow \ker h \xrightarrow{\delta} \operatorname{coker} f \longrightarrow \operatorname{coker} g \longrightarrow \operatorname{coker} h \dashrightarrow 0$$

which appears in the diagram

(continued)



Proof Here is the task list for the proof: Construct the **connecting homomorphism** δ as a map, check it is a homomorphism, show exactness of the sequence, and show naturality. We will do the first two steps, exemplify exactness at $\ker g$, and skip naturality.

To construct δ, let $c \in \ker h$. Since j surjective, there exists $b \in B$ such that $j(b) = c$ and b is unique up to adding an element $i(a)$ for some $a \in A$ by exactness at B. We have $j'(g(b)) = h(j(b)) = h(c) = 0$. Hence $g(b) \in \ker j' = \operatorname{im} i'$ by exactness at B'. Since i' is injective, there exists a unique $a' \in A'$ with $i'(a') = g(b)$. We set $\delta(c) = a' + \operatorname{im} f \in \operatorname{coker}(f)$. In this fashion, δ is a well-defined map because $g(i(a)) = i'(f(a))$, thus $f(a)$ is the unique preimage of $g(i(a))$ under i' and $f(a)$ is trivial in coker f.

To check that δ is a homomorphism, let $\delta(c_1) = a'_1 + \operatorname{im} f$, $\delta(c_2) = a'_2 + \operatorname{im} f$, and $\delta(c_1 + c_2) = a'_3 + \operatorname{im} f$ with a'_i defined via elements b'_i as above. We have to show that $a'_1 + a'_2 - a'_3 \in \operatorname{im} f$. Indeed, we have $i'(a'_1 + a'_2 - a'_3) = g(b_1 + b_2 - b_3)$ and $j(b_1 + b_2 - b_3) = c_1 + c_2 - (c_1 + c_2) = 0$. Exactness at B yields an element $a \in A$ such that $i(a) = b_1 + b_2 - b_3$. For this element, we compute

$$i'(f(a)) = g(i(a)) = g(b_1 + b_2 - b_3) = i'(a'_1 + a'_2 - a'_3).$$

Since i' is injective, it follows that $f(a) = a'_1 + a'_2 - a'_3$.

Next we show exactness at $\ker g$. Let $x \in \ker(j|_{\ker g})$. Then $x \in \ker(j) \cap \ker(g)$. By exactness at B, we have $\ker j = \operatorname{im} i$, so $x \in \operatorname{im} i$, meaning there exists $a \in A$ with $i(a) = x$. It remains to show that $a \in \ker f$. Now by injectivity of i', we have $f(a) = 0$ if and only if $i'(f(a)) = g(i(a)) = 0$. But $g(i(a)) = g(x) = 0$, so indeed $a \in \ker f$, hence $\ker(j|_{\ker g}) \subseteq \operatorname{im}(i|_{\ker f})$. Conversely, let $x \in i(\ker(f))$. Then $x = i(a)$ for some $a \in \ker f$, hence $g(x) = g(i(a)) = i'(f(a)) = i'(0) = 0$. Therefore $x \in \ker g$ but also $x \in \ker j = \operatorname{im}(i)$ by exactness at B. It follows that $x \in \ker(j|_{\ker g})$, which shows $\operatorname{im}(i|_{\ker f}) \subseteq \ker(j|_{\ker g})$. □

Recall that an exact sequence of R-modules of the form

$$0 \to A \to B \to C \to 0$$

is called a short exact sequence (SES). A **short exact sequence of chain complexes** of R-modules is a sequence of chain maps

$$0 \to A_* \to B_* \to C_* \to 0$$

which gives a short exact sequence of R-modules in every degree.

Theorem 4.4
Let $0 \to C_* \to D_* \to E_* \to 0$ be a SES of chain complexes of R-modules. Then we obtain a natural LES in homology

$$\cdots \to H_n(C_*) \longrightarrow H_n(D_*) \longrightarrow H_n(E_*) \xrightarrow{\delta} H_{n-1}(C_*) \longrightarrow H_{n-1}(D_*) \to \cdots .$$

Proof Applying the snake lemma to

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_{n+1} & \longrightarrow & D_{n+1} & \longrightarrow & E_{n+1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle c_{n+1}} & & \downarrow{\scriptstyle d_{n+1}} & & \downarrow{\scriptstyle e_{n+1}} & & \\ 0 & \longrightarrow & C_n & \longrightarrow & D_n & \longrightarrow & E_n & \longrightarrow & 0 \end{array}$$

gives that $C_n/B_n(C_*) \longrightarrow D_n/B_n(D_*) \longrightarrow E_n/B_n(E_*) \longrightarrow 0$ is exact. From

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_{n-1} & \longrightarrow & D_{n-1} & \longrightarrow & E_{n-1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C_{n-2} & \longrightarrow & D_{n-2} & \longrightarrow & E_{n-2} & \longrightarrow & 0 \end{array}$$

we get that $0 \longrightarrow Z_{n-1}(C_*) \longrightarrow Z_{n-1}(D_*) \longrightarrow Z_{n-1}(E_*)$ is exact. Hence

$$\begin{array}{ccccccccc} & & C_n/B_n(C_*) & \longrightarrow & D_n/B_n(D_*) & \longrightarrow & E_n/B_n(E_*) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z_{n-1}(C_*) & \longrightarrow & Z_{n-1}(D_*) & \longrightarrow & Z_{n-1}(E_*) & & \end{array}$$

is a diagram with exact rows. Applying the snake lemma one more time gives a six term natural exact sequence in homology with the connecting homomorphism in the middle and the remaining arrows induced by chain maps. Every six term sequence overlaps in three terms both with the predecessor and the successor. So they all unify to an infinite natural exact sequence as required. □

Let (X, A) be a pair of spaces. Applying the theorem to the SES

$$0 \longrightarrow C_*^{\text{sing}}(A; R) \longrightarrow C_*^{\text{sing}}(X; R) \longrightarrow C_*^{\text{sing}}(X, A; R) \longrightarrow 0 \qquad (4.5)$$

of singular chain complexes, we finally obtain the missing natural transformations $\partial_n : H_n^{\text{sing}} \longrightarrow H_{n-1}^{\text{sing}} \circ J$ as connecting homomorphisms and we have proven the following result.

Theorem 4.6
The pair $(H_*^{\text{sing}}, \partial_*)$ satisfies the LES axiom of a homology theory with coefficients in R-mod.

Remark 4.7
For a triple $B \subseteq A \subseteq X$, we can more generally consider the SES

$$0 \longrightarrow C_*^{\text{sing}}(A, B; R) \longrightarrow C_*^{\text{sing}}(X, B; R) \longrightarrow C_*^{\text{sing}}(X, A; R) \longrightarrow 0$$

and obtain the **natural LES of a triple**

$$\cdots \to H_n^{\text{sing}}(A, B; R) \xrightarrow{H_n^{\text{sing}}(i)} H_n^{\text{sing}}(X, B; R) \xrightarrow{H_n^{\text{sing}}(j)} H_n^{\text{sing}}(X, A; R) \xrightarrow{\partial_n} \cdots$$

where $i : (A, B) \to (X, B)$ and $j : (X, B) \to (X, A)$ are the inclusions. The LES of a pair is then just the case $B = \emptyset$.

4.3 Homotopy Invariance

To prepare the verification of the axiom of homotopy invariance, we introduce another notion from homological algebra.

Definition 4.8
An R-**chain homotopy** of R-chain maps $f_*, g_* : (C_*, c_*) \to (D_*, d_*)$ is a family of R-homomorphisms

$$h_n : C_n \to D_{n+1}$$

such that $f_n - g_n = h_{n-1} \circ c_n + d_{n+1} \circ h_n$.

We will indicate chain homotopic chain maps symbolically by $f_* \simeq_{h_*} g_*$. As the terminology suggests, we have the following lemma.

Lemma 4.9
If $f_* \simeq_{h_*} g_*$, then $H_n(f_*) = H_n(g_*)$ for all $n \in \mathbb{Z}$.

Proof For a cycle $z \in Z_n(C_*)$, the chain homotopy relation gives

$$f_n(z) = g_n(z) + h_{n-1}(c_n(z)) + d_{n+1}(h_n(z)).$$

The second summand vanishes because $c_n(z) = 0$ while the third summand is a boundary in D_*. Hence $H_n(f_*)(z + B_n(C_*)) = H_n(g_*)(z + B_n(C_*))$. □

It is easily confirmed that chain homotopy defines an equivalence relation on the set of chain maps $C_* \to D_*$. Chain homotopy is moreover compatible with composition in the sense that for two pairs of chain maps

$$f_*^1, g_*^1 \colon C_* \to D_*, \quad f_*^2, g_*^2 \colon D_* \to E_*$$

such that $f_*^1 \simeq g_*^1$ and $f_*^2 \simeq g_*^2$, we have $f_*^2 \circ f_*^1 \simeq g_*^2 \circ g_*^1$. A question that could come to one's mind is whether the converse of Lemma 4.9 holds true. So if $f_*, g_* \colon (C_*, c_*) \to (D_*, d_*)$ satisfy $H_*(f_*) = H_*(g_*)$, can we conclude $f_* \simeq g_*$? The answer is "no," and it remains "no" even if C_* and D_* are free over $\mathbb{Z}$. You will construct a counterexample in Exercise 4.1.

A chain map $f \colon (C_*, c_*) \to (D_*, d_*)$ is called a **chain homotopy equivalence** if it has a **chain homotopy inverse** $g_* \colon (D_*, d_*) \to (C_*, c_*)$, which satisfies $g_* \circ f_* \simeq \mathrm{id}_{(C_*,c_*)}$ and $f_* \circ g_* \simeq \mathrm{id}_{(D_*,d_*)}$. In this case we write $(C_*, c_*) \simeq_{f_*} (D_*, d_*)$.

Lemma 4.10
A chain homotopy equivalence $(C_*, c_*) \simeq_{f_*} (D_*, d_*)$ induces homology isomorphisms $H_*(f_*)$ in all degrees.

Proof Apply Lemma 4.9 and functoriality of $H_n \colon R\text{-chain} \to R\text{-mod}$. □

We only mention that this lemma does have a partial converse: If a chain map $f_* \colon (C_*, c_*) \to (D_*, d_*)$ of chain complexes (C_*, c_*) and (D_*, d_*) consisting of free modules over a principal ideal domain induces isomorphisms $H_*(f_*)$ in all degrees, then (C_*, c_*) and (D_*, d_*) are chain homotopy equivalent.

After these algebraic preliminaries, let us now come to the proof of homotopy invariance for singular homology. First we attack the absolute case when $A, B = \emptyset$. So let $F\colon X \times I \to Y$ be a homotopy from $F_0 = f$ to $F_1 = g$. We have to show

$$H_*^{\text{sing}}(f; R) = H_*^{\text{sing}}(g; R).$$

By Lemma 4.9, it is enough to construct a chain homotopy

$$P_n\colon C_n^{\text{sing}}(X; R) \to C_{n+1}^{\text{sing}}(Y; R)$$

from $C_*^{\text{sing}}(f, R) = f_*$ to $C_*^{\text{sing}}(g; R) = g_*$. Let $\sigma^n\colon \Delta^n \to X$ be a singular n-simplex. We obtain a composition

$$\Delta^n \times I \xrightarrow{\sigma^n \times \text{id}_I} X \times I \xrightarrow{F} Y.$$

Geometrically, we think of "$\Delta^n \times I$" as a **prism**, which we subdivide into $(n+1)$-simplices to define the **prism operator** $P_n\colon C_n^{\text{sing}}(X; R) \to C_{n+1}^{\text{sing}}(X; R)$ by

$$P_n(\sigma^n) = \sum_{i=0}^{n} (-1)^i F \circ (\sigma^n \times \text{id}_I)|_{[v_0,\ldots,v_i,w_i,\ldots,w_n]},$$

where $v_0, \ldots, v_n$ are the vertices of the "bottom" simplex $\Delta^n \times \{0\}$ and $w_0, \ldots, w_n$ are the vertices of the "top" simplex $\Delta^n \times \{1\}$. Such a subdivision is indicated in Fig. 4.1. Note that

$$[v_0, \ldots, v_i, w_i, \ldots, w_n] = [v_0, \ldots, v_i] * [w_i, \ldots, w_n],$$

is the **join** obtained by joining each point of the first space to each point of the second space by a line segment. Thus it is a non-degenerate $(n+1)$-simplex that we identify with Δ^{n+1} again by the unique linear homeomorphism that preserves the vertex order. We claim that $f_* \simeq_{P_*} g_*$ so that we have to show

$$\partial_{n+1} \circ P_n = g_n - f_n - P_{n-1} \circ \partial_n.$$

Geometrically, the formula asserts that the boundary of the prism shaped singular chain $P_n(\sigma^n)$ consists of the top simplex $g_n \circ \sigma^n$, the bottom simplex $f_n \circ \sigma^n$, and

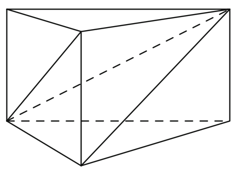


Fig. 4.1 The prism $\Delta^2 \times I$ subdivided into three 3-simplices

the vertical walls $P_{n-1}(\partial_n(\sigma^n))$. To verify the formula, we split the sum in two parts according to whether the vertex gap occurs in the bottom or the top simplex

$$\partial_{n+1}(P_n(\sigma^n)) = \sum_{0\le j\le i\le n} (-1)^i(-1)^j F\circ(\sigma^n\times \mathrm{id}_I)|_{[v_0,\ldots,\hat{v}_j,\ldots,v_i,w_i,\ldots,w_n]} + \sum_{0\le i\le j\le n} (-1)^i(-1)^{j+1} F\circ(\sigma^n\times \mathrm{id}_I)|_{[v_0,\ldots,v_i,w_i,\ldots,\hat{w}_j,\ldots,w_n]}.$$

The summands with $i = j$ cancel except for the summand with $i = 0$ from the first sum and the summand with $i = n$ from the second sum. Hence the term equals

$$F\circ(\sigma^n\times \mathrm{id}_I)|_{[w_0,\ldots,w_n]} - F\circ(\sigma^n\times \mathrm{id}_I)|_{[v_0,\ldots,v_n]} + \sum_{j<i}(\cdots) + \sum_{i<j}(\cdots).$$

The first summand equals $g_n(\sigma^n)$, the second summand equals $-f_n(\sigma^n)$, and the remaining term is equal to

$$-\left(\sum_{i<j}(-1)^i(-1)^j F\circ(\sigma^n\times \mathrm{id}_I)|_{[v_0,\ldots,v_i,w_i,\ldots,\hat{w}_j,w_n]} + \sum_{j<i}(-1)^{i-1}(-1)^j F\circ(\sigma^n\times \mathrm{id}_I)|_{[v_0,\ldots,\hat{v}_j,\ldots,v_i,w_i,\ldots,w_n]}\right)$$

which is just $-P_{n-1}(\partial_n(\sigma^n))$ separated into two lines according to whether the switch from bottom to top simplex happens before or after the vertex gap coming from the boundary homomorphism. So the proof of the claim and therefore the proof of homotopy invariance of singular homology in the absolute case are complete. For general maps of pairs $f, g\colon (X, A) \to (Y, B)$ that are homotopic through maps of pairs we obtain

$$P_n(C_n^{\mathrm{sing}}(A; R)) \subseteq C_{n+1}^{\mathrm{sing}}(B; R).$$

Thus P_n descends to a homomorphism

$$P_n\colon C_n^{\mathrm{sing}}(X, A; R) \longrightarrow C_{n+1}^{\mathrm{sing}}(Y, B; R)$$

and the relation $g_n - f_n = \partial_{n+1}\circ P_n + P_{n-1}\circ \partial_n$ remains valid because it holds for representatives. This shows that also the relative chain maps f_* and g_* induce the same homomorphism in homology and we have proven the following theorem.

Theorem 4.11
The pair $(H_*^{\mathrm{sing}}, \partial_*)$ satisfies the homotopy invariance axiom of a homology theory with coefficients in R-mod.

4.4 Excision

Recall that excision for simplicial homology as stated in Theorem 3.19 was straightforward to prove because an isomorphism can actually be implemented on the level of chain complexes. There is however no hope to obtain a similar isomorphism of singular chain complexes "$C_*^{\text{sing}}(X \setminus A, Y \setminus A) \cong C_*^{\text{sing}}(X, Y)$" because singular simplices might neither be contained in $X \setminus A$ nor in Y. Nonetheless, we can subdivide a given singular simplex in X into a singular chain that respects the decomposition $X = (X \setminus A) \cup Y$. As an outcome of this technique, we will construct a *chain homotopy equivalence* $C_*^{\text{sing}}(X \setminus A, Y \setminus A) \simeq C_*^{\text{sing}}(X, Y)$ which, as discussed in the previous section, is enough to obtain isomorphisms in homology.

To make this program precise, let us now assume more generally that our space X comes with a family of subspaces $\mathcal{U} = \{U_j\}$ such that $\bigcup_j \mathring{U}_j = X$. Let $C_n^{\mathcal{U}}(X; R) \subseteq C_n^{\text{sing}}(X; R)$ be the submodule generated by all $\sigma^n : \Delta^n \to X$ with $\sigma^n(\Delta^n) \subseteq U_j$ for some j. An element $c \in C_n^{\mathcal{U}}(X; R)$ is called a **$\mathcal{U}$-small chain**. Clearly, $C_*^{\mathcal{U}}(X; R)$ is a **subcomplex** of $C_*^{\text{sing}}(X; R)$, meaning

$$\partial(C_n^{\mathcal{U}}(X; R)) \subseteq C_{n-1}^{\mathcal{U}}(X; R)$$

for all n and the inclusion $i_* : C_*^{\mathcal{U}}(X; R) \to C_*^{\text{sing}}(X; R)$ is a chain map. Since for each j, singular simplices in U_j are $\mathcal{U}$-small, we have further inclusions of subcomplexes $C_*^{\text{sing}}(U_j) \subseteq C_*^{\mathcal{U}}(X)$ for all j.

Proposition 4.12
The inclusion $i_* : C_*^{\mathcal{U}(X;R)} \to C_*^{\text{sing}}(X; R)$ *has a chain homotopy inverse* $r_* : C_*^{\text{sing}}(X; R) \to C_*^{\mathcal{U}}(X; R)$ *such that* $r_* \circ i_* = \mathrm{id}_{C_*^{\mathcal{U}}(X;R)}$ *and such that* $i_* \circ r_* \simeq_{F_*} \mathrm{id}_{C_*^{\text{sing}}(X;R)}$ *with* $F_*(C_*^{\text{sing}}(U_j)) \subseteq C_{*+1}^{\text{sing}}(U_j)$ *for all* j.

So the chain homotopy inverse r_* makes singular chains $\mathcal{U}$-small without touching those chains that are already $\mathcal{U}$-small. Once we have proven the proposition, the excision axiom is readily verified.

Theorem 4.13
The pair $(H_*^{\text{sing}}, \partial_*)$ *satisfies the excision axiom of a homology theory with coefficients in* R-mod.

Proof Given a triple (X, Y, A) such that $\overline{A} \subseteq \mathring{Y}$, let $\mathcal{U} = \{X\backslash A, Y\}$. We have $(X\mathring{\backslash} A) \cup \mathring{Y} = (X\backslash \bar{A}) \cup \mathring{Y} = X$. So Proposition 4.12 provides us with a chain homotopy inverse

$$r_*\colon C_*^{\text{sing}}(X; R) \longrightarrow C_*^{\mathcal{U}}(X; R)$$

of the inclusion $i_*\colon C_*^{\mathcal{U}}(X; R) \to C_*^{\text{sing}}(X; R)$ such that $r_* \circ i_* = \mathrm{id}_{C_*^{\mathcal{U}}(X;R)}$ and such that $i_* \circ r_* \simeq_{F_*} \mathrm{id}_{C_*^{\text{sing}}(X;R)}$ with $F_*(C_*^{\text{sing}}(Y)) \subseteq C_{*+1}^{\text{sing}}(Y)$. Since singular chains in Y are $\mathcal{U}$-small, $C_*^{\text{sing}}(Y; R)$ is a common subcomplex of both $C_*^{\mathcal{U}}(X; R)$ and $C_*^{\text{sing}}(X; R)$. The relation $r_* \circ i_* = \mathrm{id}_{C_*^{\mathcal{U}}(X;R)}$ ensures that not only i_* but also r_* restricts to the identity on the subcomplex $C_*^{\text{sing}}(Y; R)$. Thus i_* and r_* descend to chain maps

$$C_*^{\mathcal{U}}(X; R)/C_*^{\text{sing}}(Y; R) \longleftrightarrow C_*^{\text{sing}}(X; R)/C_*^{\text{sing}}(Y; R). \tag{4.14}$$

The latter chain complex is just $C_*^{\text{sing}}(X, Y; R)$. Since $F_*(C_*^{\text{sing}}(Y)) \subseteq C_{*+1}^{\text{sing}}(Y)$, also F_* descends to a chain homotopy from $i_* \circ r_*$ to the identity on $C_*^{\text{sing}}(X, Y; R)$. Hence the two chain complexes in (4.14) are chain homotopy equivalent. The first chain complex is by definition a quotient of a non-direct sum of subcomplexes

$$C_*^{\mathcal{U}}(X; R)/C_*^{\text{sing}}(Y; R) = \left(C_*^{\text{sing}}(X \setminus A; R) + C_*^{\text{sing}}(Y; R)\right)/C_*^{\text{sing}}(Y; R),$$

hence isomorphic to $C_*^{\text{sing}}(X\backslash A; R)/C_*^{\text{sing}}(Y\backslash A; R) = C_*^{\text{sing}}(X\backslash A, Y\backslash A; R)$ because both chain complexes are free on the set of singular simplices in $X \setminus A$, which are not contained in Y. Lemma 4.10 completes the proof. □

The idea of the proof of the proposition is to construct a chain homotopy inverse r_* by subdividing simplices to make them $\mathcal{U}$-small. We already encountered the technique to do so in Sect. 3.5: **barycentric subdivision**. Recall that barycentric subdivision is defined inductively as follows. A 1-simplex is subdivided into two 1-simplices by adding the midpoint as another vertex. An n-simplex is subdivided into the $(n+1)!$ different n-simplices formed by the cones whose common tip point is the barycenter of Δ^n and whose bases are the $(n-1)$-simplices in the $(n+1)$ subdivided faces of Δ^n. As a first naive approach, we can try and define r_* by subdividing each simplex precisely as many times as necessary to make it $\mathcal{U}$-small. The problem is that this will not define a chain map in general because it can happen that one has to subdivide a given simplex to make it $\mathcal{U}$-small while it would not be necessary to subdivide the faces as is illustrated in Fig. 4.2. Therefore certain simplices could map to different chains along the two compositions in the square

$$\begin{array}{ccc} C_n^{\text{sing}}(X;R) & \xrightarrow{\partial} & C_{n-1}^{\text{sing}}(X;R) \\ \downarrow{\scriptstyle r_n} & & \downarrow{\scriptstyle r_{n-1}} \\ C_n^{\mathcal{U}}(X;R) & \xrightarrow{\partial} & C_{n-1}^{\mathcal{U}}(X;R). \end{array}$$

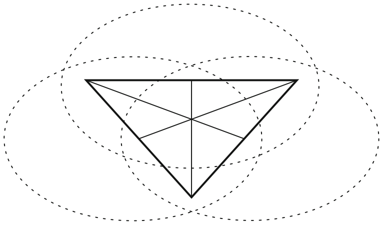


Fig. 4.2 Barycentric subdivision transforms this 2-simplex into a $\mathcal{U}$-small chain but the faces are already $\mathcal{U}$-small to begin with

So to enforce that r_* becomes a chain map, we will need to "undo" unnecessary subdivisions at the boundary. With these ideas in mind, let us now begin the formal proof of Proposition 4.12.

As a first step, we give a formal treatment of barycentric subdivision. Any $p+1$ points $v_0, \dots, v_p \in \Delta^q$ define an affine linear map $\Delta^p \to \Delta^q$. We shall allow ourselves an abuse of notation and denote this map by $\sigma = [v_0, \dots, v_p]$. Considering the q-simplex Δ^q itself as the topological space under investigation, let $L_p(\Delta^q; R) \subset C_p^{\text{sing}}(\Delta^q; R)$ be the free R-module with basis the affine linear p-simplices in Δ^q. We agree $L_p(\Delta^q; R) = 0$ for $p < 0$ so that $L_*(\Delta^q; R)$ is clearly a subcomplex of $C_*^{\text{sing}}(\Delta^q; R)$. Any fixed point $v \in \Delta^q$ defines the "cone simplex"

$$v\sigma = [v, v_0, \dots, v_p] \colon \Delta^{p+1} \longrightarrow \Delta^q.$$

Forming cones $v\sigma$ on affine linear p-simplices σ yields an R-homomorphism

$$v \colon L_p(\Delta^q; R) \longrightarrow L_{p+1}(\Delta^q; R).$$

For $\sigma = [v_0, \dots, v_p]$, let $\sigma^{\mathrm{b}} := \frac{1}{p+1}(v_0 + \cdots + v_p)$ be the barycenter of σ.

Definition 4.15

The homomorphism $B_p \colon L_p(\Delta^q; R) \to L_p(\Delta^q; R)$ defined inductively by $B_0(\sigma) = \sigma$ and $B_p(\sigma) = \sigma^{\mathrm{b}} B_{p-1}(\partial_p(\sigma))$ is called the **barycentric subdivision operator**.

So for $p = 1$ and $\sigma = [v_0, v_1]$, we obtain

$$B_1(\sigma) = \sigma^{\mathrm{b}}(B_0(\partial_1(\sigma))) = \sigma^{\mathrm{b}}(B_0([v_1] - [v_0])) = \sigma^{\mathrm{b}}([v_1] - [v_0]) =$$
$$= \left[\tfrac{1}{2}(v_0 + v_1), v_1\right] - \left[\tfrac{1}{2}(v_0 + v_1), v_0\right]$$

and for $p = 2$ and $\sigma = [v_0, v_1, v_2]$, we get accordingly

$$B_2(\sigma) = \sigma^{\mathrm{b}}(B_1 \circ \partial_2([v_0, v_1, v_2])) = \sigma^{\mathrm{b}}(B_1([v_1, v_2] - [v_0, v_2] + [v_0, v_1])) =$$
$$= \sigma^{\mathrm{b}}\Big(\left[\tfrac{1}{2}(v_1 + v_2), v_2\right] - \left[\tfrac{1}{2}(v_1 + v_2), v_1\right] - \left[\tfrac{1}{2}(v_0 + v_2), v_2\right] +$$
$$+ \left[\tfrac{1}{2}(v_0 + v_2), v_0\right] + \left[\tfrac{1}{2}(v_0 + v_1), v_1\right] - \left[\tfrac{1}{2}(v_0 + v_1), v_0\right]\Big)$$

$$= \left[\sigma^{\mathrm{b}}, \tfrac{1}{2}(v_1+v_2), v_2\right] - \left[\sigma^{\mathrm{b}}, \tfrac{1}{2}(v_1+v_2), v_1\right] - \left[\sigma^{\mathrm{b}}, \tfrac{1}{2}(v_0+v_2), v_2\right] +$$
$$+ \left[\sigma^{\mathrm{b}}, \tfrac{1}{2}(v_0+v_2), v_0\right] + \left[\sigma^{\mathrm{b}}, \tfrac{1}{2}(v_0+v_1), v_1\right] - \left[\sigma^{\mathrm{b}}, \tfrac{1}{2}(v_0+v_1), v_0\right]$$

with $\sigma^{\mathrm{b}} = \frac{1}{3}(v_0 + v_1 + v_2)$. A visualization was given in Fig. 3.4.

Lemma 4.16
Setting $B_p = 0$ for $p < 0$, the family of barycentric subdivision operators defines a chain map $B_*: L_*(\Delta^q; R) \longrightarrow L_*(\Delta^q; R)$.

Proof We show the equality $\partial_p \circ B_p = B_{p-1} \circ \partial_p$ by induction on p. For $p \leq 0$, equality is trivial. For the induction step, we observe that for $p \geq 0$, every $\sigma = [v_0, \ldots, v_p]$ and every $v \in \Delta^q$ satisfy the relation

$$\partial_{p+1}(v\sigma) = \sigma + \sum_{i=0}^{p} (-1)^{i+1} [v, v_0, \ldots, \widehat{v_i}, \ldots, v_p] = \sigma - v\partial_p\sigma.$$

Geometrically, this means the boundary of a cone simplex consists of the base simplex and the cone of the faces of the base simplex with suitable signs. Combining the formula with $\partial_{p-1} \circ B_{p-1} = B_{p-2} \circ \partial_{p-1}$ by induction hypothesis, we obtain

$$\partial_p(B_p(\sigma)) = \partial_p(\sigma^{\mathrm{b}}(B_{p-1}(\partial_p(\sigma)))) = B_{p-1}(\partial_p(\sigma)) - \sigma^{\mathrm{b}}(\partial_{p-1}(B_{p-1}(\partial_p(\sigma)))) =$$
$$= B_{p-1}(\partial_p(\sigma)) - \sigma^{\mathrm{b}}(B_{p-2}(\partial_{p-1}(\partial_p(\sigma)))) = B_{p-1}(\partial_p(\sigma))$$

because $\partial_{p-1} \circ \partial_p = 0$. □

Lemma 4.17
The diameter of the simplices in $B_p(\sigma)$ is at most $\frac{p}{p+1} \cdot \operatorname{diam}(\sigma)$.

Proof Since the simplices in $B_p(\sigma)$ are affine linearly embedded in $\Delta^q \subset \mathbb{R}^{q+1}$, the lemma has the same proof as Lemma 3.23 □

Lemma 4.18
The homomorphisms $h_*: L_*(\Delta^q; R) \to L_{*+1}(\Delta^q; R)$ defined inductively by $h_p = 0$ for $p < 0$ and $h_p(\sigma) = \sigma^b(\sigma - h_{p-1}(\partial_p(\sigma)))$ for $p \geq 0$ form a chain homotopy $\mathrm{id}_{L_*(\Delta^q;R)} \simeq_{h_*} B_*$.

Proof We show the equality $\mathrm{id}_{L_p(\Delta^q;R)} - B_p = h_{p-1} \circ \partial_p + \partial_{p+1} \circ h_p$ by induction on p. For $p < 0$, equality is trivial. For $p = 0$, equality is clear because $h_0([v]) = [v, v]$. For $p > 0$, we use the identity $\partial_{p+1}(v\sigma) = \sigma - v\partial_p\sigma$ from above to compute

$$\begin{aligned}\partial_{p+1}(h_p(\sigma)) = \partial_{p+1}(\sigma^{\mathrm{b}}(\sigma - h_{p-1}(\partial_p(\sigma)))) =&\\ = \sigma - h_{p-1}(\partial_p(\sigma)) - \sigma^{\mathrm{b}}\partial_p(\sigma - h_{p-1}(\partial_p(\sigma))) =&\\ = \sigma - h_{p-1}(\partial_p(\sigma)) - \sigma^{\mathrm{b}}\partial_p(\sigma) + \sigma^{\mathrm{b}}\partial_p(h_{p-1}(\partial_p(\sigma))).&\end{aligned}$$

By the induction hypothesis, we have

$$\begin{aligned}\partial_p h_{p-1}(\partial_p(\sigma))) = \partial_p(\sigma) - B_{p-1}(\partial_p(\sigma)) - h_{p-2}(\partial_{p-1}(\partial_p(\sigma))) =&\\ = \partial_p(\sigma) - B_{p-1}(\partial_p(\sigma)).&\end{aligned}$$

Substituting this above, we obtain

$$\begin{aligned}\partial_{p+1}(h_p(\sigma)) = \sigma - h_{p-1}(\partial_p(\sigma)) - \sigma^{\mathrm{b}}\partial_p(\sigma) + \sigma^{\mathrm{b}}\partial_p(\sigma) - \sigma^{\mathrm{b}}B_{p-1}(\partial_p(\sigma)) =&\\ = \sigma - h_{p-1}(\partial_p(\sigma)) - B_p(\sigma).&\end{aligned}$$

□

The chain homotopy h_* in degree one and two is pictured in Fig. 4.3. The homomorphism h_1 transforms each 1-simplex into a chain of three 2-simplices. The homomorphism h_2 transforms each 2-simplex into a chain of ten 3-simplices. Next we elegantly transport the barycentric subdivision procedure from affine linear chains on Δ^q to singular chains on any topological space X. To do so, we define a natural transformation $B(-)_*\colon C_*^{\mathrm{sing}}(-; R) \longrightarrow C_*^{\mathrm{sing}}(-; R)$ as follows. The component at X is defined by

$$\begin{aligned}B(X)_p\colon C_p^{\mathrm{sing}}(X; R) &\longrightarrow C_p^{\mathrm{sing}}(X; R)\\ \sigma^p &\longmapsto C_p^{\mathrm{sing}}(\sigma^p; R)(B_p(\mathrm{id}_{\Delta^p})).\end{aligned}$$

In words, we apply the functor $C_p^{\mathrm{sing}}(-; R)$ to the continuous map $\sigma^p\colon \Delta_p \to X$ and evaluate the resulting homomorphism on the barycentric subdivision of the

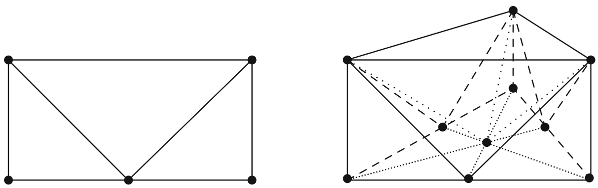


Fig. 4.3 Illustration of the chain homotopy h_* in degree one and two. To make the geometric structure visible, we shifted all occurring barycenters below their actual position. In fact, the vertical dimension is compressed to zero

singular simplex $\mathrm{id}\colon \Delta_p \to \Delta_p$. Similarly, the chain homotopy h_* gives rise to the natural transformation $h(-)_*\colon C_*^{\mathrm{sing}}(-;R) \to C_{*+1}^{\mathrm{sing}}(-;R)$ defined by

$$h(X)_p\colon C_p^{\mathrm{sing}}(X;R) \longrightarrow C_{p+1}^{\mathrm{sing}}(X;R)$$
$$\sigma^p \longmapsto C_{p+1}^{\mathrm{sing}}(\sigma^p;R)(h_p(\mathrm{id}_{\Delta^p})).$$

Lemma 4.19
The homomorphisms $B(X)_*\colon C_*^{\mathrm{sing}}(X;R) \to C_*^{\mathrm{sing}}(X;R)$ form a chain map and $\mathrm{id}_{C_*^{\mathrm{sing}}(X;R)} \simeq_{h(X)_*} B(X)_*$.

Proof In degree $p \leq 0$, the asserted relations are clear. For fixed $p > 0$, we refer to the inclusion $f_i\colon [v_0,\ldots,\widehat{v_i},\ldots,v_p] \longrightarrow [v_0,\ldots,v_p]$ of the i-th face into the standard p-simplex as the i-th **face map** of Δ_p. For $\sigma^p \in C_p^{\mathrm{sing}}(X;R)$, we have

$$\begin{aligned} B_{p-1}(X)(\partial_p\sigma^p) &= \sum_{i=0}^{p}(-1)^i C_{p-1}^{\mathrm{sing}}(\sigma^p\circ f_i)(B_{p-1}(\mathrm{id}_{\Delta_{p-1}})) = \\ &= \sum_{i=0}^{p}(-1)^i C_{p-1}^{\mathrm{sing}}(\sigma^p)(C_{p-1}^{\mathrm{sing}}(f_i)(B_{p-1}(\mathrm{id}_{\Delta_{p-1}}))) = \\ &= \sum_{i=0}^{p}(-1)^i C_{p-1}^{\mathrm{sing}}(\sigma^p)(B_{p-1}(f_i)) = C_{p-1}^{\mathrm{sing}}(\sigma^p)\left(B_{p-1}\left(\sum_{i=0}^{p}(-1)^i f_i\right)\right) \end{aligned}$$

which is just $C_{p-1}^{\mathrm{sing}}(\sigma^p)(B_{p-1}(\partial_p\,\mathrm{id}_{\Delta^p})) = \partial_p C_p^{\mathrm{sing}}(\sigma^p)(B_p(\mathrm{id}_{\Delta^p})) = \partial_p B_p(X)(\sigma^p)$ because $C_*^{\mathrm{sing}}(\sigma^p)$ is a chain map and so is B_* on affine linear chains by Lemma 4.16. Using Lemma 4.18, a similar computation shows $\mathrm{id}_{C_*^{\mathrm{sing}}(X;R)} \simeq_{h(X)_*} B(X)_*$. □

By compatibility of chain homotopy and composition, we obtain inductively that also the k-th iteration $B_*^k(X)$ of barycentric subdivision

$$B^k(X)_* \simeq B(X)_* \circ B^{k-1}(X)_* \simeq B^{k-1}(X)_* \simeq \mathrm{id}_{C_*^{\mathrm{sing}}(X;R)}$$

is chain homotopic to the identity. An explicit chain homotopy from $\mathrm{id}_{C_*^{\mathrm{sing}}(X;R)}$ to $B^k(X)_*$ is given by

$$G_*^k := \sum_{i=0}^{k-1} h(X)_* \circ B^i(X)_*$$

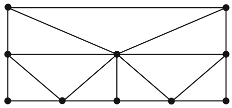


Fig. 4.4 The chain homotopy G^2_* connects a 1-simplex to its twofold subdivision. Again, in fact the vertical dimension is compressed to zero

with $B^0(X)_* = \mathrm{id}_*$ because we have the telescope sum calculation

$$\begin{aligned}\partial_{*+1}G^k_* + G^k_{*-1}\partial_* &= \sum_{i=0}^{k-1}\left(\partial_{*+1}h(X)_* \circ B^i(X)_* + h(X)_{*-1}B^i(X)_{*-1}\partial_*\right)\\ &= \sum_{i=0}^{k-1}\left(\partial_{*+1}h(X)_* + h(X)_{*-1}\partial_*\right)B^i(X)_*\\ &= \sum_{i=0}^{k-1}\left(\mathrm{id}_* - B(X)_*\right)B^i(X)_*\\ &= \sum_{i=0}^{k-1}\left(B^i(X)_* - B(X)^{i+1}_*\right) = \mathrm{id}_* - B^k(X)_*. \qquad (4.20)\end{aligned}$$

A picture of how G^2_1 acts on a 1-simplex is given in Fig. 4.4.

Lemma 4.21
For any family of subspaces $\mathcal{U} = \{U_i\}$ with $\bigcup \mathring{U}_i = X$ and for all $\sigma^p\colon \Delta^p \to X$, there exists an integer $k \geq 0$ such that $B^k(X)_p(\sigma^p)$ is $\mathcal{U}$-small.

Proof Pick a Lebesgue-δ associated with the open cover $\{(\sigma^p)^{-1}(\mathring{U}_i)\}$ of the compact metric space Δ_p. Then for any integer $k \geq 0$ with $\sqrt{2}(\frac{p}{p+1})^k < \delta$, Lemma 4.17 shows that the diameter of the simplices in the chain $B^k_p(\mathrm{id}_{\Delta^p})$ is bounded from above by δ and hence $B^k(X)_p(\sigma^p)$ is $\mathcal{U}$-small. □

After all these preliminaries, we can now finally tackle the proof of Proposition 4.12. So fix a family of subspaces $\mathcal{U} = \{U_i\}$ with $\bigcup \mathring{U}_i = X$. We define a homomorphism $F_p\colon C^{\mathrm{sing}}_p(X;R) \to C^{\mathrm{sing}}_{p+1}(X;R)$ on the basis as follows. For a given singular p-simplex σ^p, let $k(\sigma^p)$ be the minimal k as in Lemma 4.21 and set $F_p(\sigma^p) = G^{k(\sigma^p)}_p(\sigma^p)$. Using this, we define an endomorphism r_p of $C^{\mathrm{sing}}_p(X)$ by

$$r_p(\sigma^p) := B^{k(\sigma^p)}(X)_p(\sigma^p) + G^{k(\sigma^p)}_{p-1}(\partial_p(\sigma^p)) - F_{p-1}(\partial_p(\sigma^p)).$$

The formula might look intimidating but it makes the idea precise that we tried to convey in the beginning. The first summand subdivides the singular simplex σ^p as many times as necessary to make it $\mathcal{U}$-small. The second summand adds a chain of p-simplices to the boundary of σ^p such that the boundary of this chain is the difference of the undivided boundary and the $k(\sigma^p)$-times subdivided boundary of σ^p. So the second summand undoes the subdivision of the boundary produced by the first summand: The boundary of the sum of the first two terms is just the boundary of σ^p. Finally, the third summand adds a p-chain at each face of σ^p that connects the undivided face to a copy of itself that is subdivided as many times as necessary to make it $\mathcal{U}$-small.

We first observe that $r_*(C_*^{\text{sing}}(X;R)) \subseteq C_*^{\mathcal{U}}(X;R)$ because this is true for the first summand by construction and because all but the $\mathcal{U}$-small simplices cancel out in the difference $G_{*-1}^{k(\sigma^p)}(\partial_p(\sigma^p)) - F_{*-1}(\partial_p(\sigma^p))$. From (4.20), we get

$$\partial_p G_p^{k(\sigma^p)}(\sigma^p) + G_{p-1}^{k(\sigma^p)}\partial_p(\sigma^p) = \sigma^p - B^{k(\sigma^p)}(X)_p(\sigma^p).$$

Using the definition of r_p, we can rewrite this equation as

$$\partial_p F_p(\sigma^p) + F_{p-1}\partial_p(\sigma^p) = \sigma^p - r_p(\sigma^p).$$

This shows firstly that

$$\partial_p r_p(\sigma^p) = \partial_p \sigma^p - \partial_p F_{p-1}\partial_p(\sigma^p) = r_{p-1}\partial_p(\sigma^p)$$

as we see by applying the relation twice. So $r_* \colon C_*^{\text{sing}}(X;R) \longrightarrow C_*^{\mathcal{U}}(X;R)$ is a chain map. Secondly, the relation says precisely that $\mathrm{id}_{C_*^{\text{sing}}(X;R)} \simeq_{F_*} i_* \circ r_*$. Finally, we have $r_* \circ i_* = \mathrm{id}_*$ because $k(\sigma^p) = 0$ for every $\sigma^p \in C_p^{\mathcal{U}}(X;R)$ and we have $F_*(C_*^{\text{sing}}(U_j)) \subseteq C_{*+1}^{\text{sing}}(U_j)$ for all j because the $(p+1)$-chain $F_p(\sigma^p)$ has the same image in X as σ^p. This completes the proof of Proposition 4.12 and hence of Theorem 4.13.

We now turn to the dimension axiom that is by far the easiest to verify for singular homology. Actually, the singleton space will be the only space whose singular homology we will ever compute by considering the singular chain complex.

Theorem 4.22
The pair $(H_*^{\text{sing}}, \partial_*)$ satisfies the dimension axiom of a homology theory with values in R-mod *and has coefficient module $H_0(\bullet) \cong R$.*

Proof The singleton "$\bullet$" is a terminal object in the category Top. So for each $n \geq 0$, there exists precisely one map $\Delta^n \to \bullet$. Thus the singular chain complex $C_*^{\text{sing}}(\bullet; R)$ has the form

$$\begin{array}{ccccccccc} & 3 & & 2 & & 1 & & 0 & & -1 \\ \cdots \xrightarrow{\text{id}} & R & \xrightarrow{0} & R & \xrightarrow{\text{id}} & R & \xrightarrow{0} & R & \longrightarrow & 0 \end{array}$$

where the upper line indicates the degree. The differentials alternate between the trivial homomorphism and the identity homomorphism because

$$\partial(\sigma^n) = \sum_{i=0}^{n}(-1)^i \sigma^{n-1} = \begin{cases} \sigma^{n-1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}.$$

From this description the assertion is clear. □

Theorem 4.23
The pair $(H_*^{\text{sing}}(-, -; R), \partial_*)$ is an ordinary homology theory with values in R-mod and coefficient module $H_0(\bullet) \cong R$.

Proof This is just the triumphant summary of the last four theorems. □

We conclude this section with a mild generalization. Given an arbitrary R-module M, we define **singular homology with coefficients in M** by the homology $H_*^{\text{sing}}(X, A; M)$ of the chain complex $C_*^{\text{sing}}(X, A; M) := C_*^{\text{sing}}(X, A; R) \otimes_R M$ with differentials $\partial_* \otimes \text{id}_M$. Let us check that this defines an ordinary homology theory with values in R-mod and coefficient module $H_0(\bullet) \cong M$. For the dimension axiom, the given proof applies without change. For the LES axiom, one has to keep in mind that the $(-) \otimes_R M$-functor, which on morphisms acts by $(-) \otimes_R \text{id}_M$, is not necessarily left exact. So it is not a priori clear that it will turn the SES in (4.5) into a SES again. However, the SES splits by the splitting lemma (Lemma 3.18) because $C_*(X, A; R)$ is free so that we do obtain a SES after applying the functor and the rest of the proof goes through as before. In view of the chain homotopies P_* and F_* constructed above, the homotopy invariance and excision axioms follow because the $(-) \otimes_R M$-functor takes chain homotopies to chain homotopies. Since $(-) \otimes_R R$ is (naturally isomorphic to) the identity functor, we recover the old definition for $M = R$. We have thus shown existence of an ordinary homology theory with an arbitrarily prescribed coefficient module $H_0(\bullet)$. Later in the course (Theorem 6.40), we will complement this result by a uniqueness theorem for ordinary homology theories on a reasonable subcategory of $\mathsf{Top}^{(2)}$.

4.5 Singular Homology in Degree Zero and One

From the definition of singular homology it is intuitively apparent that two cycles in a space X only have a chance to be homologous if they lie in the same path component of X. Moreover, two points in the same path component define homologous 0-simplices because a path joining them can be interpreted as a 1-simplex. So the zeroth singular homology informs about the number of path components of a space. In a similar vein, after taking another glance on Fig. 3.2, it is conceivable that the first integral singular homology of a path connected space, albeit always abelian, should be related to the fundamental group. In this section, we will clarify this relationship. Let us fix an R-module M. Note that a map $f : X \to Y$ induces a map of sets $\pi_0(f) : \pi_0 X \to \pi_0 Y$ by continuity.

Theorem 4.24
We have a natural isomorphism

$$H_n^{\mathrm{sing}}(X; M) \cong \bigoplus_{X_\alpha \in \pi_0 X} H_n^{\mathrm{sing}}(X_\alpha; M).$$

Proof For all $\sigma^n \in C_n^{\mathrm{sing}}(X; R)$, the space $\sigma^n(\Delta^n)$ is path connected. Thus

$$C_n^{\mathrm{sing}}(X; R) = \bigoplus_{X_\alpha \in \pi_0 X} C_n^{\mathrm{sing}}(X_\alpha; R) \quad \text{and} \quad \partial(C_n^{\mathrm{sing}}(X_\alpha; R)) \subseteq C_{n-1}^{\mathrm{sing}}(X_\alpha; R)$$

so that we obtain the asserted decomposition already on the chain complex level. The functor $(-) \otimes_R M$ preserves this decomposition. The statement and proof of naturality are clear. □

The same proof shows that singular homology with any coefficient module preserves coproducts. Let us express this property as a separate axiom as follows.

Theorem 4.25
*Singular homology $(H_*^{\mathrm{sing}}, \partial_*)$ satisfies the **additivity axiom**: the inclusions $X_j \to X = \coprod_{i \in I} X_i$ induce isomorphisms $\bigoplus_{i \in I} H_n(X_i) \cong H_n(X)$ for all n.*

Be aware that every homology theory (H_*, ∂_*) satisfies **finite additivity**: $\bigoplus_{i=1}^k H_n(X_i) \cong H_n(\coprod_{i=1}^k X_i)$. This holds because the inclusion $i : X \to X \coprod Y$ is split injective (unless X is empty and Y is nonempty in which case we swap the roles of X and Y), so that the LES of $(X \coprod Y, X)$ splits up into split SESes

$$0 \longrightarrow H_*(X) \longrightarrow H_*(X \textstyle\coprod Y) \longrightarrow H_*(X \textstyle\coprod Y, X) \longrightarrow 0$$

and $H_*(X \coprod Y, X) \cong H_*(Y)$ by excision. So the value of the theorem is that H_n^{sing} also preserves infinite coproducts. However, it does not preserve general colimits in Top, not even general pushouts. Instead, for homotopy pushouts one obtains a long exact sequence, the **Mayer–Vietoris sequence**, as we will see later in Sect. 5.3. For short we will say a homology theory is **additive** if it satisfies the additivity axiom.

Theorem 4.26
We have an isomorphism $H_0^{\mathrm{sing}}(X; M) \cong \bigoplus_{\pi_0 X} M$.

Proof By Theorem 4.25, we only have to show $H_0^{\mathrm{sing}}(X; M) \cong M$ if X is path connected and nonempty. Consider the **augmentation homomorphism**

$$\varepsilon\colon C_0^{\mathrm{sing}}(X; R) \longrightarrow R, \quad \sum_i m_i \sigma_i^0 \longmapsto \sum_i r_i,$$

which sums the coefficients of a singular 0-chain. It is clearly surjective and we claim that $\ker \varepsilon = \operatorname{im} \partial_1$. Indeed, we have $\operatorname{im} \partial_1 \subseteq \ker \varepsilon$ because $\varepsilon(\partial_1(\sigma^1)) = \varepsilon(\sigma^1|_{[v_1]} - \sigma^1|_{[v_0]}) = 1 - 1 = 0$. To see $\ker \varepsilon \subseteq \operatorname{im} \partial_1$, let $\varepsilon\left(\sum_i r_i \sigma_i^0\right) = 0$ so that $\sum_i r_i = 0$. Fix a point $x_0 \in X$ and pick paths $\gamma_i\colon I \to X$ with $\gamma_i(0) = x_0$ and $\gamma_i(1) = \sigma_i^0(v_0)$ for all i. Each γ_i can be interpreted as a 1-simplex $\sigma_i^1\colon \Delta^1 \to X$ and we can form the singular 1-chain $\sum_i r_i \sigma_i^1$ whose singular boundary is

$$\sum_i r_i \sigma_i^1|_{[v_1]} - \sum_i r_i \sigma_i^1|_{[v_0]} = \sum_i r_i \sigma_i^1|_{[v_1]} - \left(\sum_i r_i\right) \sigma_i^1|_{[v_0]} = \sum_i r_i \sigma_i^0.$$

Thus we have shown that the sequence

$$C_1^{\mathrm{sing}}(X; R) \xrightarrow{\partial_1} C_0^{\mathrm{sing}}(X; R) \xrightarrow{\varepsilon} R \longrightarrow 0$$

is exact. Since the $(-) \otimes_R M$-functor is right exact by Exercise 3.6, the sequence remains exact after applying $(-) \otimes_R M$. This proves $H_0^{\mathrm{sing}}(X; M) \cong M$. □

Let X be a nonempty space. If we consider a path $f\colon I \to X$ as a singular 1-simplex $f\colon \Delta^1 \to X$, then this 1-simplex forms a 1-cycle if and only if the path f is a loop, meaning $f(0) = f(1)$. This observation leads to the **Hurewicz homomorphism in degree one** on which we have the following theorem.

Theorem 4.27 (Hurewicz Theorem in Degree One)
The assignment

$$\mathrm{Hur}\colon \pi_1(X, x_0) \longrightarrow H_1^{\mathrm{sing}}(X; \mathbb{Z}), \quad [f] \longmapsto f + B_1^{\mathrm{sing}}(X; \mathbb{Z})$$

yields a well-defined group homomorphism, which is natural with respect to pointed maps. If X is path connected, then Hur *descends to an isomorphism*

$$\mathrm{Hur}_{\mathrm{ab}}\colon \pi_1(X, x_0)_{\mathrm{ab}} \xrightarrow{\cong} H_1^{\mathrm{sing}}(X, \mathbb{Z}).$$

Thus $\ker \mathrm{Hur} = [\pi_1(X, x_0), \pi_1(X, x_0)]$ *is the commutator subgroup of* $\pi_1(X, x_0)$.

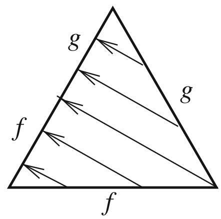


Fig. 4.5 The 2-simplex σ^2 restricts to f and g on the boundary as shown

Proof To show well-definedness, we give an alternative description of the Hurewicz homomorphism. The **fundamental class** $[S^1] \in H_1^{\text{sing}}(S^1; \mathbb{Z})$ is the homology class represented by the singular 1-cycle given by wrapping the standard 1-simplex Δ^1 once around the circle, say counter-clockwise at constant speed, starting and ending at $1 \in S^1$. We identify $I/\partial I \cong S^1$ so that given $[f] \in \pi_1(X, x_0)$, we obtain an induced map $\overline{f}\colon S^1 \to X$. Then $\text{Hur}([f]) = H_1^{\text{sing}}(\overline{f}; \mathbb{Z})([S^1])$, so Hur is well-defined as a map by Theorem 4.11.

To see that it is a homomorphism, let $[f], [g] \in \pi_1(X, x_0)$. We consider the singular 2-simplex σ^2 given by the orthogonal projection $[v_0, v_1, v_2] \to [v_0, v_2]$ composed with the map $fg\colon [v_0, v_2] \to X$ that follows the path f on the line segment from v_0 to the barycenter of $[v_0, v_2]$ and follows the path g on the line segment of $[v_0, v_2]$ from the barycenter to v_2. The map is indicated in Fig. 4.5. We have $\partial_2(\sigma^2) = g - fg + f$, so $f + g - fg$ is a 1-boundary. Identifying $[v_0, v_2]$ with the unit interval I, the pointed homotopy class $[fg]$ is just the composition of $[f]$ and $[g]$ in $\pi_1(X, x_0)$, so we have $\text{Hur}([f][g]) = \text{Hur}([f]) + \text{Hur}([g])$ as claimed.

Naturality of the Hurewicz homomorphism for a pointed map $g\colon (X, x_0) \to (Y, y_0)$ is easily verified by means of the above description of Hur. Indeed, we have

$$H_1^{\text{sing}}(g)(\text{Hur}([f]) = H_1^{\text{sing}}(g)(H_1^{\text{sing}}(\overline{f})([S^1]) = H_1^{\text{sing}}(g \circ \overline{f})([S^1]) =$$
$$= \text{Hur}([g \circ f]) = \text{Hur}(\pi_1(g)([f])).$$

Now assume X is path connected. To see that Hur_{ab} is an isomorphism, pick paths γ_x from x_0 to x for all $x \in X$. Using these, we define a homomorphism $\psi\colon C_1^{\text{sing}}(X; \mathbb{Z}) \to \pi_1(X, x_0)_{\text{ab}}$ of abelian groups on the generating 1-simplices by $\psi(f) = [\gamma_{f(v_0)}\, f\, \overline{\gamma_{f(v_1)}}]_{\text{ab}}$. We verify that ψ descends to a homomorphism $\overline{\psi}\colon H_1^{\text{sing}}(X; \mathbb{Z}) \to \pi_1(X, x_0)_{\text{ab}}$. So let $\sigma^2\colon \Delta^2 = [v_0, v_1, v_2] \to X$ be a singular 2-simplex. We have to show that $\psi(\partial\sigma^2) = 0$. Let f, g, and h be the zeroth, first, and second face of σ^2. Then $\psi(\partial_2(\sigma^2)) = \psi(f - g + h) = \psi(f) - \psi(g) + \psi(h)$ is represented by the concatenation $\gamma_{\sigma^2(v_1)}\, f\, \overline{\gamma_{\sigma^2(v_2)}}\gamma_{\sigma^2(v_2)}\, \overline{g}\, \overline{\gamma_{\sigma^2(v_0)}}\gamma_{\sigma^2(v_0)}\, h\, \overline{\gamma_{\sigma^2(v_1)}}$, which is pointed homotopic to $\gamma_{\sigma^2(v_1)}\, f\, \overline{g}\, h\, \overline{\gamma_{\sigma^2(v_1)}}$ and hence pointed null-homotopic because the loop $f\, \overline{g}\, h$ can be shrunk to the constant loop at v_1 through the interior of σ^2. Hence $\psi(\partial_2\sigma^2) = 0$ as required. We claim that $\overline{\psi}$ is the inverse of Hur_{ab}. Indeed, on the one hand, we have

$$\overline{\psi}(\text{Hur}_{\text{ab}}([f]_{\text{ab}})) = \overline{\psi}(f + B_1) = [\gamma_{f(v_0)}\, f\, \overline{\gamma_{f(v_1)}}]_{\text{ab}} = [\gamma_{x_0}]_{\text{ab}} + [f]_{\text{ab}} - [\gamma_{x_0}]_{\text{ab}} = [f]_{\text{ab}}$$

for all $[f]_{\text{ab}} \in \pi_1(X, x_0)_{\text{ab}}$ with $B_1 = B_1^{\text{sing}}(X; \mathbb{Z})$. Next we observe that the assignment sending $x \in X$ to the 1-simplex γ_x extends to a homomorphism $\gamma\colon C_0^{\text{sing}}(X, \mathbb{Z}) \to$

$C_1^{\text{sing}}(X;\mathbb{Z})$. For a singular 1-chain $c \in C_1^{\text{sing}}(X;\mathbb{Z})$, we then have $\text{Hur}_{\text{ab}} \circ \psi(c) = c + \gamma(\partial c) + B_1$. It follows that on the other hand

$$\text{Hur}_{\text{ab}}(\overline{\psi}(z + B_1)) = z + \gamma(\partial z) + B_1 = z + \gamma(0) + B_1 = z + B_1$$

for all $z + B_1 \in H_1^{\text{sing}}(X;\mathbb{Z})$. □

In the example $(X, x_0) = (S^1, 1)$, let $f : I \to S^1$ be the generator of $\pi_1(X, x_0) = \pi_1(X, x_0)_{\text{ab}}$ given by $f(t) = \exp(2\pi \mathrm{i} t)$. We then have

$$\text{Hur}(f) = H_1^{\text{sing}}(\text{id}_{S^1};\mathbb{Z})([S^1]) = \text{id}_{H_1^{\text{sing}}(S^1;\mathbb{Z})}([S^1]) = [S^1],$$

so the fundamental class $[S^1]$ is actually a generator of the infinite cyclic group $H_1^{\text{sing}}(S^1;\mathbb{Z})$. Similarly, we have fundamental classes $[S^n]$ generating $H_n^{\text{sing}}(S^n;\mathbb{Z})$ for all $n > 1$. Correspondingly, we obtain the higher Hurewicz homomorphisms $\text{Hur}_n : \pi_n(X, x_0) \to H_n^{\text{sing}}(X;\mathbb{Z})$ given by $\text{Hur}_n(f) = H_n^{\text{sing}}(f;\mathbb{Z})([S^n])$. We just inform the reader that for these, one can prove the **Hurewicz theorem in higher degrees**, which asserts that if $\pi_k(X, x_0)$ is trivial for $k < n$, then Hur_n is an isomorphism and Hur_{n+1} is surjective.

Witold Hurewicz, born 1904 in Łódź, was an influential Polish–American mathematician. He proved the Hurewicz theorem on the first homology group as well as the generalization to higher homotopy groups. While being a brilliant thinker, the "Dictionary of Scientific Biography" [6] also describes him as notoriously absentminded and suggests that this trait might have been the reason for the sudden end of his life. On an excursion during the International Symposium on Algebraic Topology at the National Autonomous University of Mexico, he tragically fell off the top of a Mayan step pyramid in Uxmal on the peninsula of Yucatán on September, 4th, 1956 and died two days later in a hospital in Mérida from his severe injuries [17]. His mathematics, however, will live forever on.

Exercises

4.1 This exercise is based on an example appearing in [10, p. 106]. Consider the following diagram in Ab (equivalently $\mathbb{Z}$-mod).

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow {\scriptstyle \binom{1}{0}} & & \downarrow {\scriptstyle \text{id}} & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{(2\ 1)} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

(a) Explain that the rows define chain complexes (C_*, c_*) and (D_*, d_*) and that the vertical arrows form a chain map $\phi_*\colon (C_*, c_*) \to (D_*, d_*)$.
(b) Show that $H_n(\phi_*) = 0$ for all $n \in \mathbb{Z}$.
(c) Show that ϕ_* is not chain homotopic to the zero chain map.

4.2 A chain complex (C_*, c_*) of R-modules is called **exact** (also **acyclic**) if $\ker c_n = \operatorname{im} c_{n+1}$ for all $n \in \mathbb{Z}$. It is called **contractible** if the identity chain map id_{C_*} is chain homotopic to the zero chain map. Let C_* be the exact chain complex given by a short exact sequence of R-modules filled up with trivial modules in all remaining degrees. Show that C_* is contractible if and only if the short exact sequence splits.

4.3 Show that setting $h_n(X, A) := \prod_k H_k^{\mathrm{sing}}(X, A; \mathbb{Z}) \,/\, \bigoplus_k H_k^{\mathrm{sing}}(X, A; \mathbb{Z})$ independent of n with boundary map $\partial_*\colon h_n(X, A) \to h_{n-1}(A)$ induced from the singular differential in the apparent way defines a homology theory satisfying $h_n(\bullet) = 0$ for all $n \in \mathbb{Z}$. Show that (h_*, ∂_*) is not additive and hence in particular not zero.

4.4 Give an example of a nontrivial covering map $Y \to X$ with $H_0^{\mathrm{sing}}(Y; \mathbb{Z}) \cong \mathbb{Z}$ and $H_1^{\mathrm{sing}}(X; \mathbb{Z}) = 0$.

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Now that we have verified the Eilenberg–Steenrod axioms for singular homology and analyzed what it has to say in low degrees, the rules of the game are to avoid any more arguments with simplices. Instead, we derive conclusions from the axioms only, to make sure that the results will be valid for all homology theories. So until further notice, in this chapter (H_*, ∂_*) denotes a homology theory with values in R-mod.

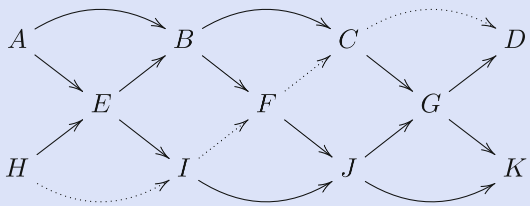
5.1 Relative vs. Absolute Homology

We motivated the definition of relative homology $H_n(X, A)$ in the simplicial context of Sect. 3.3 by saying that sometimes one wants to disregard the homology produced by the subspace A. However, it seems that the same can be achieved by just considering the absolute homology $H_n(X/A)$ of the quotient space X/A in which A is collapsed to point. So the question arises if we have an isomorphism $H_n(X/A) \cong H_n(X, A)$. The answer is "typically yes," but "generally no." In fact, we will show that the answer is yes if the homology theory (H_*, ∂_*) is ordinary, if (X, A) is a cofibration, and if $n \geq 1$.

To start, recall that in Remark 4.7 we encountered the triple sequence of singular homology. We want to see that this more general long exact homology sequence is not intrinsic to singular homology but can in fact be deduced from the axioms of any homology theory. The key argument is another diagram chase.

H. Kammeyer, *Introduction to Algebraic Topology*, Compact Textbooks in Mathematics, https://doi.org/10.1007/978-3-030-98313-0_5

Lemma 5.1 (Braid Lemma)
Consider the commutative diagram



consisting of four interwoven sequences of R-modules. If each of the three solid sequences is exact and the dotted sequence is a chain complex, then also the dotted sequence is exact.

Proof It will hardly cost more effort to do the chase yourself than to read this proof. Every object in the diagram is the domain of two arrows, an upper one and a lower one. To refer to these arrows, we write u for the upper and l for the lower arrow and we add the domain as an index. By the chain complex condition on the dotted sequence, we only need to verify the "ker $\subseteq$ im" inclusion at I, F, and C.

Exactness at I. Let $i \in \ker u_I$. Then $l_I(i) = l_F(u_I(i)) = l_F(0) = 0$, hence there exists $e \in E$ with $l_E(e) = i$. Since $l_B(u_E(e)) = u_I(l_E(e)) = u_I(i) = 0$, there exists $a \in A$ such that $u_A(a) = u_E(e)$. Thus $u_E(e - l_A(a)) = u_E(e) - u_A(a) = 0$. Therefore, there exists $h \in H$ such that $u_H(h) = e - l_A(a)$. It follows that $l_H(h) = l_E(u_H(h)) = l_E(e - l_A(a)) = i - l_E(l_A(a)) = i$.

Exactness at F. Let $f \in \ker u_F$. Then $u_J(l_F(f)) = l_C(u_F(f)) = l_C(0) = 0$, so there exists $i \in I$ with $l_I(i) = l_F(f)$. Thus we have $l_F(f - u_I(i)) = l_F(f) - l_F(u_I(i)) = l_F(f) - l_I(i) = l_F(f) - l_F(f) = 0$. This shows that there exists $b \in B$ such that $l_B(b) = f - u_I(i)$. We have $u_B(b) = u_F(l_B(b)) = u_F(f - u_I(i)) = -u_F(u_I(i)) = 0$, so there exists $e \in E$ with $u_E(e) = b$. Thus we have $u_I(l_E(e) + i) = u_I(l_E(e)) + u_I(i) = l_B(u_E(e)) + u_I(i) = l_B(b) + u_I(i) = f - u_I(i) + u_I(i) = f$.

Exactness at C. Let $c \in \ker u_C$. Then $u_G(l_C(c)) = 0$, so there exists $j \in J$ with $u_J(j) = l_C(c)$. We have $l_J(j) = l_G(u_J(j)) = l_G(l_C(c)) = 0$, thus we obtain $f \in F$ such that $l_F(f) = j$. Since $l_C(c - u_F(f)) = l_C(c) - l_C(u_F(f)) = l_C(c) - u_J(l_F(f)) = l_C(c) - u_J(j) = l_C(c) - l_C(c) = 0$, there exists $b \in B$ such that $u_B(b) = c - u_F(f)$. This shows that $u_F(l_B(b) + f) = u_F(l_B(b)) + u_F(f) = u_B(b) + u_F(f) = c - u_F(f) + u_F(f) = c$. ◻

Theorem 5.2 (Triple Sequence)
For every triple (X, A, B) of spaces, we have a natural LES

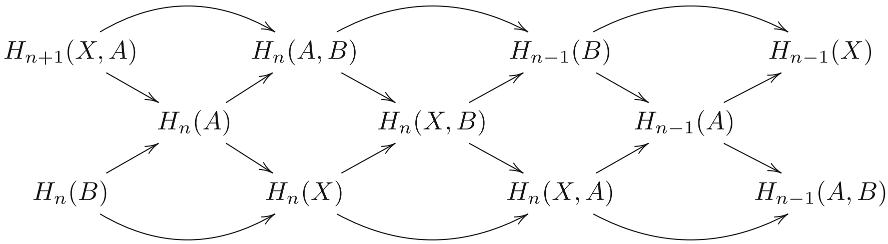
$$\cdots \longrightarrow H_n(A,B) \xrightarrow{H_n(i)} H_n(X,B) \xrightarrow{H_n(j)} H_n(X,A) \xrightarrow{\partial_n} H_{n-1}(A,B) \longrightarrow \cdots$$

where $i\colon (A, B) \to (X, B)$ and $j\colon (X, B) \to (X, A)$ denote the inclusions.

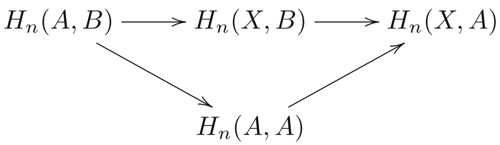
Proof We define the boundary homomorphism ∂_n by the composition

$$H_n(X, A) \xrightarrow{\partial(X,A)} H_{n-1}(A) \xrightarrow{H_{n-1}(l)} H_{n-1}(A, B)$$

with the inclusion $l\colon (A, \emptyset) \to (A, B)$ so that we get a commutative braid diagram



in which three strings are exact, to wit the LESes of (A, B), (X, B), and (X, A). The fourth string is the triple sequence of interest. In this sequence, the composition of two arrows is zero. That is clear in two of the three relevant cases because the composition factors through a composition in one of the exact sequences by commutativity. The remaining composition is trivial because we have the factorization



and the LES of (A, A) has the form

$$\cdots \to H_n(A) \xrightarrow{\text{id}} H_n(A) \xrightarrow{0} H_n(A, A) \xrightarrow{0} H_{n-1}(A) \xrightarrow{\text{id}} H_{n-1}(A) \to \cdots$$

so that $H_n(A, A) = 0$. We conclude from the braid lemma that the triple sequence is exact. Naturality follows from naturality of ∂_* and functoriality of H_*. □

The category $\mathsf{Top}_\bullet$ of pointed (hence nonempty) spaces has a zero object: The one point space "$\bullet$" whose only point also serves as base point. Our homology theory H_* restricts to a family of functors on $\mathsf{Top}_\bullet$, taking the value $H_*(X, x_0)$ on the pointed space (X, x_0). We observe the welcome property that the zero object $(\bullet, \bullet)$ has zero homology $H_*(\bullet, \bullet) = 0$ in all degrees as is immediate from the LES of the pair $(\bullet, \bullet)$. When dealing with unpointed but exclusively nonempty spaces X, it is often desirable to have a slight modification $\widetilde{H}_*(X)$ of the absolute homology $H_*(X)$ that would behave similarly as homology relative to a point, so that we still have $\widetilde{H}_*(\bullet) = 0$. But $\widetilde{H}_*$ ought to be functorial so that no choices of base points are allowed. Therefore we agree on the following definition.

Definition 5.3

The **reduced homology** of a nonempty space X is given by

$$\widetilde{H}_n(X) = \ker(H_n(X) \longrightarrow H_n(\bullet))$$

where $H_n(X) \to H_n(\bullet)$ is induced by the unique map $X \to \bullet$.

To see that reduced homology is indeed a functor, we observe that every map $f: X \to Y$ gives rise to a commutative triangle

$$\begin{array}{ccc} H_n(X) & \xrightarrow{H_n(f)} & H_n(Y) \\ & \searrow \quad \swarrow & \\ & H_n(\bullet) & \end{array}$$

so that $H_n(f)$ maps the kernel of the left arrow to the kernel of the right arrow and we can define $\widetilde{H}_n(f)$ by the restriction of $H_n(f)$ to the kernel of the left arrow. It then clearly remains true that $\widetilde{H}_n(f) = \widetilde{H}_n(g)$ if $f \simeq g$ and we see that $\widetilde{H}_n(\bullet) = \ker \mathrm{id}_{H_n(\bullet)} = 0$ for all $n \in \mathbb{Z}$. Consequently, we have $\widetilde{H}_n(f) = 0$ for all $n \in \mathbb{Z}$ if $f: X \to Y$ is **null-homotopic** (homotopic to a constant map) because in the homotopy category HoTop, we obtain a factorization

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \quad \nearrow & \\ & \bullet & \end{array}$$

To clarify the exact relationship between reduced and unreduced homology, let us now pick a base point $x_0 \in X$. We obtain the map $j: (X, \emptyset) \to (X, x_0)$ and the retraction $r: X \to \{x_0\}$, so the LES of (X, x_0) consists of split SESes

$$0 \longrightarrow H_n(\{x_0\}) \underset{H_n(r)}{\overset{\longrightarrow}{\longleftarrow}} H_n(X) \xrightarrow{H_n(j)} H_n(X, x_0) \longrightarrow 0.$$

We saw in the proof of Lemma 3.18 that $H_n(j)$ restricts to an isomorphism on $\ker H_n(r) = \widetilde{H}_n(X)$, so we obtain isomorphisms

$$\widetilde{H}_n(X) \cong H_n(X, x_0) \quad \text{and} \quad H_n(X) \cong \widetilde{H}_n(X) \oplus H_n(\bullet) \tag{5.4}$$

for all $n \in \mathbb{Z}$ and these are natural with respect to pointed maps. If H_* is ordinary, the latter isomorphism takes the form

$$\left\{ \begin{array}{l} H_0(X) \cong \widetilde{H}_0(X) \oplus H_0(\bullet), \\ H_n(X) \cong \widetilde{H}_n(X) \text{ for } n \neq 0 \end{array} \right\} \tag{5.5}$$

Now we are finally prepared to spell out the exact relationship between relative homology and absolute homology of the collapse space as promised at the beginning of this section.

Proposition 5.6
Let (X, A) be a cofibration. Then the collapse map

$$q\colon (X, A) \longrightarrow (X/A, A/A)$$

induces isomorphisms

$$H_*(X, A) \xrightarrow{\cong} H_*(X/A, A/A) \cong \widetilde{H}_*(X/A)$$

Proof Recall from Example 1.39 that the mapping cone C_i of $i\colon A \subseteq X$ is constructed out of the mapping cylinder M_i by the pushout

$$\begin{array}{ccc} A & \longrightarrow & \bullet \\ \downarrow & & \downarrow \\ M_i & \longrightarrow & C_i \end{array}$$

where the left hand arrow is the inclusion into the end of the cylinder with coordinate one. By Theorem 1.42 (ii), we thus have a homeomorphism $M_i \setminus A \cong C_i \setminus \bullet$ and hence a homeomorphism of pairs $(M_i \setminus A, A \times [0, 1)) \cong (C_i \setminus \bullet, CA \setminus \bullet)$. The strong deformation retraction H from Example 1.38 restricts to a strong deformation retraction on $M_i \setminus A$, showing that $(X, A) \simeq (M_i \setminus A, A \times [0, 1))$. The collapse map q thus factorizes as

$$\begin{array}{ccc} (X, A) & \xrightarrow{\quad q \quad} & (X/A, A/A) \\ \simeq \downarrow & & \uparrow c \; \simeq \\ (C_i \setminus \bullet, CA \setminus \bullet) & \longrightarrow & (C_i, CA) \end{array}$$

and the vertical arrows are homotopy equivalences, the right hand one being the comparison map from Theorem 2.22 corresponding to the pushout (1.37) defining X/A. Applying the functor H_*, we thus see that $H_*(q)$ factorizes through isomorphisms by homotopy invariance and excision. □

Note that the proposition remains true if A is empty in which case it asserts $H_*(X) \cong \widetilde{H}_*(X \coprod \bullet)$. In the next theorem, however, we have to assume $A \neq \emptyset$.

Theorem 5.7
Let (X, A) be a nonempty cofibration. We have a natural LES

$$\cdots \longrightarrow \widetilde{H}_n(A) \longrightarrow \widetilde{H}_n(X) \longrightarrow \widetilde{H}_n(X/A) \xrightarrow{\partial} \widetilde{H}_{n-1}(A) \longrightarrow \cdots$$

Proof Pick $a_0 \in A$ and use (5.4) and Proposition 5.6 to replace relative homology in the triple sequence of (X, A, a_0) with reduced homology. □

A pointed space (X, x_0) is called **well-pointed** if (X, x_0) is a cofibration. The condition is fairly mild but recall Exercise 2.6 for a non-well-pointed space. If (X, x_0) is any pointed space and (Y, y_0) is well-pointed, then Theorem 2.21 (i) shows that $(X \vee Y, Y)$ is a cofibration. The inclusion $i_Y : Y \to X \vee Y$ has the collapse map $q_Y : X \vee Y \to X \vee Y / X \cong Y$ as a left inverse. Hence the splitting lemma (Lemma 3.18) implies that the LES in reduced homology of $(X \vee Y, X)$ consists of split SESes and the product maps $(\widetilde{H}_*(q_X), \widetilde{H}_*(q_Y))$ provide isomorphisms

$$\widetilde{H}_*(X \vee Y) \xrightarrow{\cong} \widetilde{H}_*(X) \oplus \widetilde{H}_*(Y) \tag{5.8}$$

whose inverses are the coproduct maps $\widetilde{H}_*(i_X) \oplus \widetilde{H}_*(i_Y)$. Inductively, it follows that reduced homology preserves finite coproducts in the category $\mathsf{Top}_\bullet^{\text{well}}$ of well-pointed spaces. A little later (Theorem 5.31) we shall see that if H_* satisfies the dimension axiom, then $\widetilde{H}_*$ also preserves infinite coproducts in $\mathsf{Top}_\bullet^{\text{well}}$.

Theorem 5.7 gives moreover rise to the so-called **suspension isomorphism** in homology as we discuss next. The **suspension** of a space X is the space $SX = CX / X$ where $X \subset CX$ embeds by the base inclusion. Alternatively, the suspension can be thought of as the pushout

$$\begin{array}{ccc} X & \longrightarrow & CX \\ \downarrow & & \downarrow \\ CX & \longrightarrow & SX \end{array}$$

so that the two cone tips form the "mounting points" of the suspension. For example suspending the $(n-1)$-sphere we obtain the n-sphere, $S\,S^{n-1} = S^n$. The double meaning of the letter "S" thus becomes a fortunate notational collision. To be more rigorous, we have $S\,S^{n-1} = CS^{n-1}/S^{n-1}$, which can be canonically identified with D^n/S^{n-1}. We then identify D^n/S^{n-1} with S^n by means of the homeomorphism

$$u_n : D^n/S^{n-1} \xrightarrow{\cong} S^n \tag{5.9}$$
$$[(tx_1, \ldots, tx_n)] \longmapsto (ux_1, \ldots, ux_n, 2t - 1)$$

where $t \in [0,1]$, $(x_1, \ldots, x_n) \in S^{n-1}$ and $u = \sqrt{1-(2t-1)^2}$. Geometrically, this map sends rays from the center of the disk to meridians from the south pole to the north pole. These remarks make the identification $SS^{n-1} = S^n$ precise. The identification also holds for $n = 0$ noting that $S^{-1} = \emptyset$, thus $CS^{-1} = \bullet$ so that $SS^{-1} = S^0$ where we agree that $u_0 \colon D^0/S^{-1} \to S^0$ is the bijection sending the additional base point in $D^0/\emptyset$ to the base point $-1 \in S^0$. Suspension clearly defines a functor $S\colon \mathsf{Top} \to \mathsf{Top}$.

Theorem 5.10
We have a natural isomorphism $\widetilde{H}_{n+1}(SX) \cong \widetilde{H}_n(X)$ for every nonempty space X and all $n \in \mathbb{Z}$.

Proof We saw in Example 2.18 that (CX, X) is an NDR pair. The boundary maps in the corresponding LES of Theorem 5.7 give the desired natural isomorphisms because CX is contractible. □

As an application, we obtain the homology of the spheres just from the axioms.

Corollary 5.11
We have $H_k(D^n, S^{n-1}) \cong \widetilde{H}_k(S^n) \cong H_{k-n}(\bullet)$ and hence

$$H_k(S^n) \cong H_{k-n}(\bullet) \oplus H_k(\bullet)$$

Assuming the dimension axiom, this gives

$$H_k(D^n, S^{n-1}) \cong \begin{cases} H_0(\bullet) & \text{if } k = n, \\ 0 & \text{otherwise,} \end{cases} \qquad H_k(S^n) \cong \begin{cases} H_0(\bullet) & \text{if } k = 0 \text{ or } k = n, \\ 0 & \text{otherwise} \end{cases}$$

Proof By Example 2.18, the pair (D^n, S^{n-1}) is an NDR, so Proposition 5.6 and the homeomorphism in (5.9) show that $H_k(D^n, S^{n-1}) \cong \widetilde{H}_k(S^n)$. The suspension isomorphism in Theorem 5.10 gives $\widetilde{H}_k(S^n) \cong \widetilde{H}_{k-n}(S^0) \cong H_{k-n}(\bullet)$. □

The suspension functor S has a reduced version Σ on $\mathsf{Top}_\bullet$ defined by collapsing the arc $x_0 \times I \subset SX$ of the base point. If (X, x_0) is well-pointed, then the inclusion $x_0 \times I \subset SX$ is a cofibration (Exercise 5.2). Hence Corollary 2.25 shows that the collapse map is a homotopy equivalence $SX \simeq \Sigma X$, which is clearly natural for pointed maps. Thus we also have a natural suspension isomorphism

$$\widetilde{H}_{n+1}(\Sigma X) \cong \widetilde{H}_n(X) \tag{5.12}$$

for well-pointed spaces. Reduced suspension is left adjoint to the **loop space** functor Ω on $\mathsf{Top}_\bullet$ given by $\Omega(X, x_0) = \mathrm{Hom}_{\mathsf{Top}_\bullet}((S^1, \bullet), (X, x_0))$ with the compact open topology and the constant loop as base point. The adjunction relation

$$\mathrm{Hom}_{\mathsf{Top}_\bullet}(\Sigma(X, x_0), (Y, y_0)) \cong \mathrm{Hom}_{\mathsf{Top}_\bullet}((X, x_0), \Omega(Y, y_0))$$

can be visualized as follows. Given a map $f : \Sigma(X, x_0) \to (Y, y_0)$ and a point $x \in X$, we can restrict f to the arc $x \times I$ in $\Sigma(X, x_0)$ to obtain a loop in (Y, y_0) and thus a point in $\Omega(Y, y_0)$. The adjunction descends to $\mathsf{HoTop}_\bullet$ where in the special case $(X, x_0) = (S^n, \bullet)$, it gives the natural isomorphism

$$\pi_{n+1}(Y, y_0) \cong \pi_n(\Omega(Y, y_0)) \tag{5.13}$$

One may view the suspension isomorphism (5.12) and the "adjoint" loop space isomorphism (5.13) as a first indication of the intricate interrelation of homology and homotopy, giving an idea as to why the suspension and loop space constructions become all the more important the farther one advances in the theory.

5.2 Simplicial and Singular Homology Agree

The purpose of this section is to show that singular and simplicial homology coincide on Δ-complexes. For simplicity, we will drop the coefficient ring R from our notation. So $H_*^{\mathrm{sing}}(X, A)$ means $H_*^{\mathrm{sing}}(X, A; R)$ and $H_n^{\Delta}(X, A)$ means $H_n^{\Delta}(X, A; R)$. Since $(\Delta, \partial\Delta_n)$ and (D^n, S^{n-1}) are homeomorphic pairs, we just saw in the last corollary that $H_n^{\mathrm{sing}}(\Delta^n, \partial\Delta^n) \cong R$ is free of rank one. It is an important additional information that the homology class of the relative cycle id_{Δ^n} is a generator.

Proposition 5.14
We have $H_n^{\mathrm{sing}}(\Delta^n, \partial\Delta^n) \cong R$, *generated by the class of* id_{Δ^n}.

Proof This is clear for $n = 0$. Assuming the assertion for $n - 1$, we now argue that it also holds true for n. Let $\Lambda = (\partial\Delta^n)\backslash[v_1, \dots, v_n]$ be the 0-th **horn**. Then

$$H_n^{\mathrm{sing}}(\Delta^n, \partial\Delta^n) \cong H_{n-1}^{\mathrm{sing}}(\partial\Delta^n, \Lambda) \tag{5.15}$$

$$\cong \widetilde{H}_n^{\mathrm{sing}}(\partial\Delta^n/\Lambda) \tag{5.16}$$

$$\cong \widetilde{H}_{n-1}^{\mathrm{sing}}(\Delta^{n-1}/\partial\Delta^{n-1}) \tag{5.17}$$

$$\cong H_{n-1}^{\mathrm{sing}}(\Delta^{n-1}, \partial\Delta^{n-1}) \tag{5.18}$$

Here (5.15) holds by the triple sequence of $(\Delta^n, \partial\Delta^n, \Lambda)$, since $(\Delta^n, \Lambda) \simeq (\Lambda, \Lambda)$. Step (5.16) holds by Proposition 5.6 because the horn $\Lambda \subset \Delta^n$ is both a strong deformation retract and the zero set of a continuous function to I, so it is a (trivial) cofibration by the characterization in Theorem 2.15 (iii). Step (5.17) is true because the spaces are homeomorphic and (5.18) follows again from Proposition 5.6 because $(\Delta^{n-1}, \partial\Delta^{n-1}) \cong (D^{n-1}, S^{n-2})$ is an NDR. The singular simplex id_{Δ^n} maps to the chain $\partial_n(\mathrm{id}_{\Delta^n})$ along (5.15) which is relatively homologous to the inclusion $\Delta^{n-1} \to [v_1, \dots, v_n]$ of the 0-th face. Following the remaining isomorphisms, it thus maps to $\mathrm{id}_{\Delta^{n-1}}$, which generates $H_{n-1}^{\mathrm{sing}}(\Delta^{n-1}, \partial\Delta^{n-1})$ by induction assumption. So the relative cycle id_{Δ^n} generates $H_n^{\mathrm{sing}}(\Delta^n, \partial\Delta^n)$. □

Before we can conclude the equality of simplicial and singular homology from this proposition, we need another diagram chase to obtain the following convenient tool from homological algebra. You are once more strongly advised to do the chase yourself and only look at the proof in case you get stuck.

Lemma 5.19 (Five Lemma)
Suppose the diagram

$$\begin{array}{ccccccccc}
A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{l} & E \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\varepsilon} \\
A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{l'} & E'
\end{array}$$

in R-mod *has exact rows, where* α *is surjective,* ϵ *is injective, and* β *and* δ *are isomorphisms. Then* γ *is an isomorphism, too.*

Proof To see that γ is injective, let $c \in \ker\gamma$. Then $\delta(k(c)) = k'(\gamma(c)) = k'(0) = 0$ and since δ is injective, we have $k(c) = 0$. By exactness of the upper row, there is $b \in B$ with $j(b) = c$. We have $j'(\beta(b)) = \gamma(j(b)) = \gamma(c) = 0$ so that by exactness of the lower row, we obtain $a' \in A'$ with $i'(a') = \beta(b)$. Since α is surjective, there exists $a \in A$ with $\alpha(a) = a'$. We compute $\beta(i(a) - b) = i'(\alpha(a)) - \beta(b) = 0$. As β is injective, it follows that $i(a) = b$, thus $c = j(b) = j(i(a)) = 0$ because $\mathrm{im}\, i = \ker j$.

To see that γ is surjective, let $c' \in C'$. Since δ is surjective, there exists $d \in D$ with $\delta(d) = k'(c')$. Since $l'(k'(c')) = 0$ by exactness at D', the commutativity of the right most square gives $\varepsilon(l(d)) = 0$, so $l(d) = 0$ because ε is injective. Exactness at D provides an element $c \in C$ such that $k(c) = d$. We have $k'(\gamma(c) - c') = \delta(k(c)) - k'(c') = \delta(d) - k'(c') = 0$, so exactness at C' gives an element $b' \in B'$ with $j'(b') = \gamma(c) - c'$. Since β is surjective, there exists $b \in B$ such that $\beta(b) = b'$. Using this, we finally obtain $\gamma(c - j(b)) = \gamma(c) - \gamma(j(b)) = \gamma(c) - j'(\beta(b)) = \gamma(c) - j'(b') = \gamma(c) - \gamma(c) + c' = c'$. □

Now let (X, A) be a Δ-pair. We have a canonical homomorphism

$$H_n^{\Delta}(X, A) \longrightarrow H_n^{\mathrm{sing}}(X, A)$$

induced by the chain map

$$C_n^{\Delta}(X, A) \longrightarrow C_n^{\text{sing}}(X, A) \tag{5.20}$$

which views an n-simplex $\sigma_\alpha^n \colon \Delta_n \to X$ in the Δ-complex X as a singular simplex.

Theorem 5.21
The homomorphisms $H_n^{\Delta}(X, A) \longrightarrow H_n^{\text{sing}}(X, A)$ are isomorphisms for all n and all Δ-pairs (X, A).

Proof First suppose that $A = \emptyset$. Let $X^k \subseteq X$ be the **k-skeleton** of X consisting of all simplices of dimension at most k. We obtain a commutative ladder of LESes

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H_n^{\Delta}(X^{k-1}) & \longrightarrow & H_n^{\Delta}(X^k) & \longrightarrow & H_n^{\Delta}(X^k, X^{k-1}) & \longrightarrow \cdots \\ & \downarrow & & \downarrow & & \downarrow & \\ \cdots \longrightarrow & H_n^{\text{sing}}(X^{k-1}) & \longrightarrow & H_n^{\text{sing}}(X^k) & \longrightarrow & H_n^{\text{sing}}(X^k, X^{k-1}) & \longrightarrow \cdots \end{array}$$

We show that the right vertical map is an isomorphism. For the underlying relative simplicial chain complex we obtain

$$C_n^{\Delta}(X^k, X^{k-1}) \cong \begin{cases} \bigoplus_{\sigma_\alpha^k} R\sigma_\alpha^k & \text{if } n = k, \\ 0 & \text{otherwise} \end{cases}$$

Therefore all differentials in the chain complex are zero so that the homology is equal to the chain modules,

$$H_n^{\Delta}(X^k, X^{k-1}) \cong \begin{cases} \bigoplus_{\sigma_\alpha^k} R\sigma_\alpha^k & \text{if } n = k, \\ 0 & \text{otherwise} \end{cases}$$

Let $\Phi = \coprod_\alpha \sigma_\alpha^k$ be the unique map from the coproduct $\coprod_{\sigma_\alpha^k} \Delta^k$ to X through which each σ_α^k factors. The map Φ becomes a homeomorphism after collapsing the boundaries of the k-simplices as indicated in the diagram

$$\begin{array}{ccc} (\coprod_\alpha \Delta_\alpha^k, \coprod_\alpha \partial\Delta_\alpha^k) & \xrightarrow{\Phi} & (X^k, X^{k-1}) \\ \downarrow q & & \downarrow q' \\ (\coprod_\alpha \Delta_\alpha^k / \coprod_\alpha \partial\Delta_\alpha^k, \bullet) & \xrightarrow{\cong} & (X^k/X^{k-1}, \bullet). \end{array}$$

The pair $(\coprod \Delta_\alpha^k, \coprod \partial\Delta_\alpha^k)$ is an NDR and so is (X^k, X^{k-1}) by Theorem 2.21 (ii). Indeed, property (iv) in the definition of Δ-complexes effects by Lemma A.1 that $(\coprod_\alpha \Delta_\alpha^k) \coprod X^{k-1} \longrightarrow X^k$ is an identification map, so

$$\begin{array}{ccc} \coprod_\alpha \partial\Delta_\alpha^k & \longrightarrow & X^{k-1} \\ \downarrow & & \downarrow \\ \coprod_\alpha \Delta_\alpha^k & \longrightarrow & X^k \end{array}$$

is a pushout square. It thus follows from Proposition 5.6 that $H_*^{\text{sing}}(\Phi)$ is an isomorphism in relative homology. Additivity gives isomorphisms $H_*^{\text{sing}}(\coprod_\alpha \Delta_\alpha^k) \cong \bigoplus_\alpha H_*^{\text{sing}}(\Delta_\alpha^k)$ and $H_*^{\text{sing}}(\coprod_\alpha \partial\Delta_\alpha^k) \cong \bigoplus_\alpha H_*^{\text{sing}}(\partial\Delta_\alpha^k)$. Mapping the LES of the pair $(\coprod_\alpha \Delta_\alpha^k, \coprod_\alpha \partial\Delta_\alpha^k)$ to the direct sum over all α of the LESes of $(\Delta_\alpha^k, \partial\Delta_\alpha^k)$, the five lemma thus gives us isomorphisms

$$H_*^{\text{sing}}(\textstyle\coprod_\alpha \Delta_\alpha^k, \coprod_\alpha \partial\Delta_\alpha^k) \cong \bigoplus_\alpha H_*^{\text{sing}}(\Delta_\alpha^k, \partial\Delta_\alpha^k)$$

and by Proposition 5.14 the summands are generated by $\mathrm{id}_{\Delta_\alpha^k}$. So the morphism

$$H_n^\Delta(X^k, X^{k-1}) \longrightarrow H_n^{\text{sing}}(X^k, X^{k-1})$$

of free R-modules restricts to a bijection of generating sets whence is an isomorphism. Therefore applying the five lemma to the commutative ladder above gives the step of an induction showing $H_n^\Delta(X^k) \cong H_n^{\text{sing}}(X^k)$ for all $k \geq 0$ where the beginning $k = 0$ is clear. Since $H_n^\Delta(X) \cong H_n^\Delta(X^k)$ for $k > n$, we have

$$H_*^\Delta(X) \cong \operatorname{colim}_k H_*^\Delta(X^k)$$

Using that a singular simplex $\sigma\colon \Delta^n \to X$ has compact image, one sees that every singular n-chain in X already lies in some k-skeleton X^k. This implies that we also have $H_*^{\text{sing}}(X) \cong \operatorname{colim}_k H_*^{\text{sing}}(X^k)$ from which we conclude

$$H_*^\Delta(X) \cong \operatorname{colim}_k H_*^\Delta(X^k) \cong \operatorname{colim}_k H_*^{\text{sing}}(X^k) \cong H_*^{\text{sing}}(X)$$

The general case $A \neq \emptyset$ follows again from the five lemma. □

Corollary 5.22
Suppose a pair of spaces (X, A) admits a Δ-pair structure with finitely many n-simplices outside A and assume R is a principal ideal domain. Then $H_n^{\text{sing}}(X, A; R)$ is finitely presented.

Proof Over a principal ideal domain, submodules of finite rank free modules are finite rank free. By assumption we have that $C_n^{\Delta}(X, A; R)$ is free of finite rank, thus so is the submodule $Z_n^{\Delta}(X, A; R)$ and so is the submodule $B_n^{\Delta}(X, A; R)$. The above theorem completes the proof. □

Recall the definition of the chain map $C_*^{\Delta}(f)$ for a simplicial map of simplicial pairs $f: (X, A) \to (Y, B)$ given in Sect. 3.5. Since $C_*^{\text{sing}}(f)$ is always given by composing simplices with f, the chain map (5.20) is not natural in (X, A). But one checks that the induced map in homology is. So the isomorphisms from Theorem 5.21 are natural in the simplicial setting. More precisely, if $\mathcal{F}: \mathsf{Simp}^{(2)} \to \mathsf{Top}^{(2)}$ is the forgetful functor, then the functors

$$H_n^{\Delta},\ H_n^{\text{sing}} \circ \mathcal{F}: \mathsf{Simp}^{(2)} \to R\text{-}\mathsf{mod}$$

are naturally isomorphic. By simplicial approximation (Theorem 3.22), it follows that any continuous map $f: (X, A) \to (Y, B)$ of simplicial pairs induces a well-defined morphism $H_n^{\Delta}(X^{[r]}, A^{[r]}) \to H_n^{\Delta}(Y, B)$ on any fine enough barycentric subdivision of X. If for an R-module M we define $H_*^{\Delta}(X, A; M)$ by the homology of $C_*^{\Delta}(X, A; R) \otimes_R M$ as in the case of singular homology, then Theorems 5.21 and naturality still hold for singular and simplicial homology with coefficients in M. The proof of Corollary 5.22, however, only carries over if one requires that M is finite rank free.

5.3 The Mayer–Vietoris Sequence

The Mayer–Vietoris sequence is to homology as van Kampen's theorem is to the fundamental group. It allows to work out the homology of a space by decomposing it into smaller spaces with known homology. Such a decomposition can either be a suitable cover by two subspaces or a homotopy pushout. In either case, we derive yet another LES that under favorable circumstances computes the homology of the space from the homology of its pieces. Still (H_*, ∂_*) denotes a homology theory with values in R-mod.

Definition 5.23

Let X be the union of two subspaces $X_1, X_2 \subseteq X$. Then we call $(X; X_1, X_2)$ an **excisive triad** if the inclusion $(X_1, X_1 \cap X_2) \to (X, X_2)$ induces isomorphisms $H_n(X_1, X_1 \cap X_2) \xrightarrow{\cong} H_n(X, X_2)$ for all $n \in \mathbb{Z}$.

It turns out that a triad $(X; X_1, X_2)$ is excisive if and only if $(X; X_2, X_1)$ is excisive, but this is not completely trivial and involves some diagram chasing [29, Proposition 10.7.1]. If X_1 and X_2 are open, then $(X; X_1, X_2)$ is excisive by the excision axiom applied to the subspaces $A = X \backslash X_1$ and $Y = X_2$.

Theorem 5.24 (Mayer–Vietoris Sequence)
Let $(X; X_1, X_2)$ be an excisive triad and let $A \subseteq X_1 \cap X_2 =: X_0$ be a subspace. Then we have a natural LES

$$\to H_n(X_0, A) \xrightarrow{(H_n(i_1), H_n(i_2))} H_n(X_1, A) \oplus H_n(X_2, A) \xrightarrow{H_n(j_1) - H_n(j_2)} H_n(X, A) \xrightarrow{\partial}$$

where $i_k : (X_0, A) \to (X_k, A)$ and $j_k : (X_k, A) \to (X, A)$ are the inclusions.

Proof The triples (X_1, X_0, A) and (X, X_2, A) give rise to a commutative ladder

$$\begin{array}{ccccccccc}
\longrightarrow & H_n(X_0, A) & \xrightarrow{H_n(i_1)} & H_n(X_1, A) & \xrightarrow{H_n(l_1)} & H_n(X_1, X_0) & \xrightarrow{\partial'} & H_{n-1}(X_0, A) & \longrightarrow \\
& \downarrow{\scriptstyle H_n(i_2)} & & \downarrow{\scriptstyle H_n(j_1)} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle H_{n-1}(i_2)} & \\
\longrightarrow & H_n(X_2, A) & \xrightarrow{H_n(j_2)} & H_n(X, A) & \xrightarrow{H_n(l_2)} & H_n(X, X_2) & \xrightarrow{\partial'} & H_{n-1}(X_2, A) & \longrightarrow
\end{array}$$

by naturality of the triple sequence. We define the morphism ∂ by the composition

$$H_n(X, A) \xrightarrow{H_n(l_2)} H_n(X, X_2) \xleftarrow{\cong} H_n(X_1, X_0) \xrightarrow{\partial'} H_{n-1}(X_0, A)$$

appearing in the diagram. Naturality of the differential in the Mayer–Vietoris LES follows then likewise from naturality of triple sequences and is clear for the other arrows. Exactness is the usual diagram chasing:

Exactness at $H_n(X_0, A)$. Take an element $z \in H_n(X_0, A)$ with $H_n(i_1)(z) = 0$. By exactness, there exists $z_1 \in H_{n+1}(X_1, X_0)$ with $\partial'(z_1) = z$. If in addition $H_n(i_2)(z) = 0$ holds, then viewing z_1 as an element of $H_{n+1}(X, X_2)$ by the isomorphism, we have $\partial'(z_1) = 0$ by commutativity, hence exactness in the lower sequence gives $z_2 \in H_{n+1}(X, A)$ with $H_{n+1}(l_2)(z_2) = z_1$, so $\partial(z_2) = \partial'(z_1) = z$. This shows $\ker(H_n(i_1), H_n(i_2)) \subseteq \operatorname{im} \partial$. Suppose $z \in H_n(X_0, A)$ has a preimage under ∂. Then z also has a preimage under ∂', so $H_n(i_1)(z) = 0$ by exactness of the upper sequence. Similarly, $H_n(i_2)(z)$ has a preimage under $\partial' \circ H_n(l_2)$ but this composition is zero by exactness of the lower sequence whence $H_n(i_2)(z) = 0$. This shows $\operatorname{im} \partial \subseteq \ker(H_n(i_1), H_n(i_2))$.

Exactness at $H_n(X_1, A) \oplus H_n(X_2, A)$. Suppose the pair $(z_1, z_2) \in H_n(X_1, A) \oplus H_n(X_2, A)$ satisfies $H_n(j_1)(z_1) = H_n(j_2)(z_2)$. Then $H_n(l_1)(z_1)$ maps under the isomorphism to $H_n(l_2)(H_n(j_1)(z_1)) = H_n(l_2)(H_n(j_2)(z_2)) = 0$ by exactness in the lower sequence. Hence, there exists $z \in H_n(X_0, A)$ such that $H_n(i_1)(z) = z_1$. Moreover, we have $H_n(j_2)(H_n(i_2)(z) - z_2) = H_n(j_1)(H_n(i_1)(z)) - H_n(j_1)(z_1) = 0$. By exactness in the lower sequence, there exists $z_3 \in H_{n+1}(X, X_2)$ such that $\partial'(z_3) = H_n(i_2)(z) - z_2$. Viewing $z_3 \in H_{n+1}(X_1, X_0)$ via the isomorphism, we have $H_n(i_1)(z - \partial'(z_3)) = H_n(i_1)(z) = z_1$ by exactness of the upper sequence. In addition, we have $H_n(i_2)(z - \partial'(z_3)) = H_n(i_2)(z) - (H_n(i_2)(z) - z_2) = z_2$. This shows $\ker(H_n(j_1) - H_n(j_2)) \subseteq \operatorname{im}(H_n(i_1), H_n(i_2))$. The reverse inclusion is immediate from commutativity of the first square in the above ladder.

Exactness at $H_n(X, A)$. For $z_1 \in H_n(X_1, A)$ we have $(\partial \circ H_n(j_1))(z_1) = (\partial' \circ H_n(l_1))(z_1) = 0$ because the latter is a composition of two maps in a LES. Similarly, $(\partial \circ H_n(j_2))(z_2) = 0$ for $z_2 \in H_n(X_2, A)$ because the composition involves a composition of two arrows in a LES right away. Thus also every difference $H_n(j_1)(z_1) - H_n(j_2)(z_2)$ lies in $\ker \partial$. This shows $\operatorname{im}(H_n(j_1) - H_n(j_2)) \subseteq \ker \partial$. Let $z \in \ker \partial$. Then $H_n(l_2)(z)$ followed by the (reversed) isomorphism gives an element $z_1 \in \ker \partial' = \operatorname{im} H_n(l_1)$. Thus there is $z_2 \in H_n(X_1, A)$ with $H_n(l_1)(z_2) = z_1$. Let $z_3 = H_n(j_1)(z_2)$. Then $H_n(l_2)(z - z_3) = 0$ because both z and z_3 map to z_1 under the reversed isomorphism. Therefore by exactness $z - z_3 \in \operatorname{im} H_n(j_2)$. Let $z_4 \in H_n(X_2, A)$ be an element with $H_n(j_2)(z_4) = z - z_3$. Then $(H_n(j_1) - H_n(j_2))(z_2 \oplus (-z_4)) = z_3 + (z - z_3) = z$. This shows $\ker \partial \subseteq \operatorname{im}(H_n(j_1) - H_n(j_2))$. □

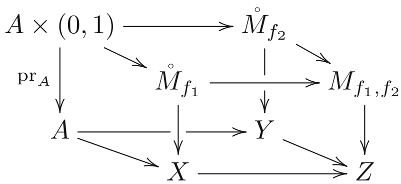
Theorem 5.25 (Mayer–Vietoris Sequence for Homotopy Pushouts)
Let

$$\begin{array}{ccc} A & \xrightarrow{f_2} & Y \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle g_2} \\ X & \xrightarrow{g_1} & Z \end{array}$$

be a homotopy pushout. Then we have a natural LES

$$\to H_n(A) \xrightarrow{(H_n(f_1),\, H_n(f_2))} H_n(X) \oplus H_n(Y) \xrightarrow{H_n(g_1) - H_n(g_2)} H_n(Z) \xrightarrow{\partial} H_{n-1}(A) \to$$

Proof From Sects. 2.3 and 2.4, we have a homotopy commutative diagram



(5.26)

in which all downward pointing arrows are homotopy equivalences. The upper square is the pushout of an open cover, so the triad $(M_{f_1,f_2}; \mathring{M}_{f_1}, \mathring{M}_{f_2})$ is excisive. Therefore the upper row in the induced diagram

$$\begin{array}{ccccccc} \xrightarrow{\partial} H_n(A \times (0,1)) & \longrightarrow & H_n(\mathring{M}_{f_1}) \oplus H_n(\mathring{M}_{f_2}) & \longrightarrow & H_n(M_{f_1,f_2}) \xrightarrow{\partial} \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ H_n(A) & \longrightarrow & H_n(X) \oplus H_n(Y) & \longrightarrow & H_n(Z) \end{array}$$

is exact by the previous theorem. If one homotopy pushout maps to another to form a homotopy commutative cube, we obtain an up to homotopy defined map of the double mapping cylinders preserving the open covers. This shows naturality. □

Recall from Sect. 2.3 that our main example of a homotopy pushout is a pushout

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow{\overline{f}} & Z. \end{array} \tag{5.27}$$

in Top where i is a cofibration. Then also j is a cofibration by Theorem 2.21 (i) so that all front pointing arrows in diagram 5.26 are inclusions and the diagram commutes strictly. Since the triad $(M_{f_1,f_2}; \mathring{M}_{f_1}, \mathring{M}_{f_2})$ is excisive, also the pushout (5.27) is excisive in the sense that the map $\overline{f}$ induces relative homology isomorphisms

$$H_n(X, A) \cong H_n(Z, Y) \tag{5.28}$$

Using these, we can prove a relative version of the Mayer–Vietoris sequence for the pushout (5.27).

Theorem 5.29
Consider the pushout (5.27) in which i (hence j) is a cofibration. Let $B \subseteq A$ and $B' \subseteq Y$ be subsets with $f(B) \subseteq B'$. Then we have a natural LES

$$\to H_n(A, B) \xrightarrow{(H_n(i),\, H_n(f))} H_n(X, B) \oplus H_n(Y, B') \xrightarrow{H_n(\overline{f}) - H_n(j)} H_n(Z, B') \to .$$

Proof Similarly as in the proof of Theorem 5.24, the boundary homomorphism ∂ is now contained in the diagram

$$\begin{array}{ccccccccc} \longrightarrow & H_n(A,B) & \xrightarrow{H_n(i)} & H_n(X,B) & \longrightarrow & H_n(X,A) & \xrightarrow{\partial'} & H_{n-1}(A,B) & \longrightarrow \\ & {\scriptstyle H_n(f)}\downarrow & & {\scriptstyle H_n(\overline{f})}\downarrow & & {\scriptstyle H_n(\overline{f})}\downarrow{\scriptstyle \cong} & & {\scriptstyle H_{n-1}(f)}\downarrow & \\ \longrightarrow & H_n(Y,B') & \xrightarrow{H_n(j)} & H_n(Z,B') & \longrightarrow & H_n(Z,Y) & \xrightarrow{\partial'} & H_{n-1}(Y,B') & \longrightarrow \end{array}$$

and exactness and naturality of the LES follow exactly as above. □

If $X_1 \cap X_2$ is not empty in Theorem 5.24, we choose $A = \{x_0\}$ for some $x_0 \in X_1 \cap X_2$ and get the **reduced Mayer–Vietoris LES for excisive triads**

$$\cdots \longrightarrow \widetilde{H}_n(X_0) \xrightarrow{(\widetilde{H}_n(i_1),\, \widetilde{H}_n(i_2))} \widetilde{H}_n(X_1) \oplus \widetilde{H}_n(X_2) \xrightarrow{\widetilde{H}_n(j_1) - \widetilde{H}_n(j_2)} \widetilde{H}_n(X) \xrightarrow{\partial} \cdots$$

For a map $f\colon (X; X_1, X_2) \to (Y; Y_1, Y_2)$ of excisive triads, we pick $x_0 \in X_1 \cap X_2$ and use $y_0 = f(x_0) \in Y_1 \cap Y_2$ to identify reduced homology with homology relative to a point so that naturality in Theorem 5.24 and naturality of the isomorphism (5.4) gives naturality of the reduced Mayer–Vietoris LES for excisive triads. Replacing homology with reduced homology in the proof of Theorem 5.25, we thus also obtain a natural **reduced Mayer–Vietoris LES for homotopy pushouts**

$$\cdots \longrightarrow \widetilde{H}_n(A) \xrightarrow{(\widetilde{H}_n(f_1),\, \widetilde{H}_n(f_2))} \widetilde{H}_n(X) \oplus \widetilde{H}_n(Y) \xrightarrow{\widetilde{H}_n(g_1) - \widetilde{H}_n(g_2)} \widetilde{H}_n(Z) \xrightarrow{\partial} \cdots$$

Example 5.30 Let us see what happens if we apply the reduced Mayer–Vietoris sequence to the homotopy pushout given by the cell attaching pushout in Top

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{f} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ D^n & \xrightarrow{\overline{f}} & Z \end{array}$$

from Example 1.40. Since $\widetilde{H}_*(D^n) = 0$ throughout, the sequence takes the form

$$\to \widetilde{H}_*(S^{n-1}) \xrightarrow{\widetilde{H}_*(f)} \widetilde{H}_*(Y) \xrightarrow{\widetilde{H}_*(j)} \widetilde{H}_*(Z) \xrightarrow{\partial} \widetilde{H}_{*-1}(S^{n-1}) \xrightarrow{\widetilde{H}_{*-1}(f)} \widetilde{H}_{*-1}(Y) \to .$$

We see that $\widetilde{H}_k(j)$ is an isomorphism, meaning the k-th reduced homology of Y remains unchanged when attaching an n-cell, if and only if $\widetilde{H}_k(f)$ is trivial and $\widetilde{H}_{k-1}(f)$ is injective. If (H_*, ∂_*) satisfies the dimension axiom, this is automatic whenever $k \notin \{n, n-1\}$. Hence attaching an n-cell only has an effect on ordinary homology in degree n and $n-1$. Here the relevant portion of the Mayer–Vietoris sequence says that the n-th homology can only grow while the $(n-1)$-st homology can only diminish when attaching an n-cell. In fact, the n-th homology remains unchanged if and only if $\widetilde{H}_{n-1}(f)$ is injective and the $(n-1)$-st homology is unaffected if and only if $\widetilde{H}_{n-1}(f)$ is trivial. These are mutually exclusive options (unless $H_0(\bullet) = 0$). Our observations are in accordance with the rough intuition that ordinary homology "counts holes in a space": The n-cell can introduce a new "$(n+1)$-dimensional hole," which for example always happens if f is null-homotopic. On the other hand, the n-cell can cover an n-dimensional hole if f attaches the cell along the $(n-1)$-dimensional "boundary" of the hole.

We conclude this section with the verification that additivity of H_* for topological sums implies additivity of $\widetilde{H}_*$ for wedge sums.

Theorem 5.31
Suppose that the homology theory (H_*, ∂_*) *satisfies the additivity axiom and let* $(X_i, x_i)_{i\in I}$ *be any family of well-pointed spaces. Then the inclusions* $X_i \to \bigvee_{j\in I} X_j$ *induce an isomorphism*

$$\bigoplus_{i\in I} \widetilde{H}_*(X_i) \cong \widetilde{H}_*\left(\bigvee_{i\in I} X_i\right)$$

Proof Combining the additivity axiom with the five lemma, we obtain

$$\bigoplus_{i\in I} \widetilde{H}_i(X_i) \cong \bigoplus_{i\in I} H_*(X_i, x_i) \cong H_*\left(\coprod_{i\in I} X_i, \coprod_{i\in I}\{x_i\}\right)$$

Since all (X_i, x_i) are well-pointed, the left hand arrow of the pushout

$$\begin{array}{ccc} \coprod_{i\in I}\{x_i\} & \longrightarrow & \bullet \\ \downarrow & & \downarrow \\ \coprod_{i\in I} X_i & \longrightarrow & \bigvee_{i\in I} X_i \end{array}$$

is a cofibration, so the isomorphism in (5.28) gives

$$H_*\left(\coprod_{i\in I} X_i, \coprod_{i\in I}\{x_i\}\right) \cong H_*\left(\bigvee_{i\in I} X_i, \bullet\right) \cong \widetilde{H}_*\left(\bigvee_{i\in I} X_i\right)$$

□

Leopold Vietoris, born 1891 in Bad Radkersburg, was an Austrian mathematician not only known for his contributions to topology but also for being a supercentenarian. With a life span of 110 years and 309 days, he still remains the oldest confirmed male Austrian that has ever lived [12]. Vietoris married his first wife Klara Riccabona in 1928 [14]. The couple had six children, all of them girls [27]. Klara died from childbed fever after giving birth to their sixth daughter [28] in 1935. In 1936, Vietoris married Klara's sister Maria [13]. Vietoris passed away in April 2002, only a few days after Maria's death at the age of 100 years and 249 days. He wrote his last article [31] on trigonometric sums at the age of 103.

5.4 Degree

According to our computation in Corollary 5.11, we have $\widetilde{H}_n^{\text{sing}}(S^n;\mathbb{Z}) \cong \mathbb{Z}$. Therefore the endomorphism ring of $\widetilde{H}_n^{\text{sing}}(S^n;\mathbb{Z})$ is canonically isomorphic to $\mathbb{Z}$ so that we can define an interesting invariant of self-maps $f\colon S^n \to S^n$ of the n-sphere.

Definition 5.32
The **degree** of a map $f\colon S^n \longrightarrow S^n$ is the unique integer $\deg f \in \mathbb{Z}$ such that $\widetilde{H}_n^{\text{sing}}(f;\mathbb{Z})(z) = \deg f \cdot z$ for all $z \in \widetilde{H}_n^{\text{sing}}(S^n;\mathbb{Z})$.

A couple of observations on the degree of a map $f\colon S^n \longrightarrow S^n$ are immediate:

- If $f \simeq g$, then $\deg f = \deg g$.
- If f is null-homotopic, then $\deg f = 0$ (hence $\deg f = 0$ if f is not surjective).
- The composition of $f, g\colon S^n \to S^n$ satisfies $\deg(f \circ g) = \deg f \cdot \deg g$.
- If f is a homotopy equivalence, then $\deg f = \pm 1$.
- We have $\deg(\mathrm{id}_{S^n}) = 1$.

Not at all obvious is however the fact that multiplication with $\deg f$ also describes the induced endomorphism for any other reduced homology theory.

Theorem 5.33
Let (H_*, ∂) be any homology theory with values in R-mod and let $f\colon S^n \to S^n$ be any map. Then the endomorphism $\widetilde{H}_n(f)$ of $\widetilde{H}_n(S^n)$ is given by multiplication with the integer $\deg f$.

This result will be key for a uniqueness theorem in the next chapter showing that the Eilenberg–Steenrod axioms determine *ordinary* homology on a large class of spaces. The proof of Theorem 5.33 will occupy the rest of this section though we allow ourselves to include an amusing application along the way. So still let (H_*, ∂) denote any homology theory with values in R-mod (ordinary or not).

Proposition 5.34
For $n \geq 0$, let $r_n\colon S^n \to S^n$ be the reflection through the coordinate plane $\{x_{n+1} = 0\} \subset \mathbb{R}^{n+1}$ so that $r_n(x_1, \ldots, x_{n+1}) = (x_0, \ldots, x_n, -x_{n+1})$. Then we have $\widetilde{H}_n(r_n) = -\mathrm{id}_{\widetilde{H}_n(S^n)}$ and hence in particular $\deg r_n = -1$.

Proof We prove the proposition by induction on n. To begin, we note that r_0 defines a map of excisive triads

$$r_0\colon (S^0; \{-1\}, \{+1\}) \longrightarrow (S^0; \{+1\}, \{-1\})$$

Consider the inclusions $i_{-1}\colon \{-1\} \to S^0$ and $i_{+1}\colon \{+1\} \to S^0$ as well as the unique map $s\colon \{-1\} \to \{+1\}$, which induces a canonical isomorphism $\sigma = H_0(s)$ from $H_0(\{-1\})$ to $H_0(\{+1\})$. Naturality of the Mayer–Vietoris sequence in Theorem 5.24 yields a commutative square of isomorphisms

$$\begin{array}{ccc} H_0(\{-1\}) \oplus H_0(\{+1\}) & \xrightarrow{H_0(i_{-1})-H_0(i_{+1})} & H_0(S^0) \\ \tau\downarrow & & \downarrow H_0(r_0) \\ H_0(\{-1\}) \oplus H_0(\{+1\}) & \xrightarrow{H_0(i_{+1})-H_0(i_{-1})} & H_0(S^0) \end{array}$$

with $\tau(a \oplus b) = \sigma^{-1}(b) \oplus \sigma(a)$. Let $D \subseteq H_0(\{-1\}) \oplus H_0(\{+1\})$ be the diagonal submodule consisting of all elements $a \oplus \sigma(a)$ for $a \in H_0(\{-1\})$. The upper isomorphism $H_0(i_{-1}) - H_0(i_{+1})$ restricts to an isomorphism $\delta\colon D \to \widetilde{H}_0(S^0)$ on D whereas τ restricts to id_D on D. Thus $\widetilde{H}_0(r_0) = (-\delta) \circ \mathrm{id}_D \circ \delta^{-1} = -\mathrm{id}_{\widetilde{H}_0(S^0)}$.

For the induction step, let $\eta\colon \widetilde{H}_n(S(S^{n-1})) \cong \widetilde{H}_{n-1}(S^{n-1})$ be the natural suspension isomorphism from Theorem 5.10. We identify $r_n = S(r_{n-1})$ to see that

$$\widetilde{H}_n(r_n) = \widetilde{H}_n(S(r_{n-1})) = \eta^{-1} \circ \widetilde{H}_{n-1}(r_{n-1}) \circ \eta = \eta^{-1} \circ (-\mathrm{id}) \circ \eta = -\mathrm{id}$$

by induction assumption. □

Note that any two reflections of $\mathbb{R}^{n+1}$ along n-dimensional linear subspaces differ by a rotation and thus are homotopic. So the reflection hyperplane in the above result is as good as any other.

Theorem 5.35
For any map $f\colon S^{2n} \to S^{2n}$, there is $x \in S^{2n}$ with $f(x) = \pm x$.

Proof Otherwise there are homotopies $\mathrm{id}_{S^{2n}} \simeq_F f$ and $f \simeq_G -\mathrm{id}_{S^{2n}}$ given by

$$F(x,t) = \frac{(1-t)x + tf(x)}{\|(1-t)x + tf(x)\|},$$

$$G(x,t) = \frac{(1-t)f(x) + t(-x)}{\|(1-t)f(x) + t(-x)\|}$$

hence $\mathrm{id}_{S^{2n}} \simeq -\mathrm{id}_{S^{2n}}$. But $-\mathrm{id}_{S^{2n}}$ is the composition of the $2n+1$ reflections through the $2n+1$ coordinate hyperplanes of $\mathbb{R}^{2n+1}$, hence $1 = \deg(\mathrm{id}_{S^{2n}}) = \deg(-\mathrm{id}_{S^{2n}}) = (-1)^{2n+1} = -1$, which is absurd. □

Theorem 5.36 (Hairy Ball Theorem)
A continuous tangential vector field on an even dimensional sphere has a zero: If $v\colon S^{2n} \to \mathbb{R}^{2n+1}$ *satisfies* $\langle v(x), x\rangle = 0$ *for all* $x \in S^{2n}$*, then there is* $x_0 \in S^{2n}$ *with* $v(x_0) = 0$.

Proof If not, then $f\colon S^{2n} \to S^{2n}$, $x \mapsto \frac{v(x)}{\|v(x)\|}$ has $x_0 \in S^{2n}$ with $f(x_0) = \pm x_0$, so

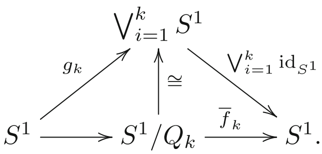
$$0 = \frac{\langle v(x_0), x_0\rangle}{\|v(x_0)\|} = \langle f(x_0), x_0\rangle = \langle \pm x_0, x_0\rangle = \pm 1$$

which again is absurd. □

The theorem says that one cannot comb a hairy ball without leaving a cowlick. Also, wind direction on earth at a fixed time is an example of a continuous tangential vector field on S^2. Thus at any time there exists a windless location on earth. In the next proposition, consider the circle S^1 as the complex numbers of modulus one.

Proposition 5.37
For $k \in \mathbb{Z}$*, the map* $f_k\colon S^1 \to S^1$ *given by* $f_k(z) = z^k$ *satisfies* $\widetilde{H}_1(f_k) = k \cdot \mathrm{id}_{\widetilde{H}_1(S^1)}$ *and hence in particular* $\deg f_k = k$.

Proof By means of complex conjugation, we write $f_{-k}(z) = \overline{f_k(z)} = r_1(f_k(z))$, so Proposition 5.34 enables us to assume $k \geq 0$. Let $Q_k = \{z \in S^1 \colon z^k = 1\}$ be the set of k-th roots of unity. Then the map f_k factors as either of the outer compositions in the diagram



The vertical homeomorphism carries the circle in the bouquet S^1/Q_k parameterized by $t \mapsto [e^{\frac{2\pi i t}{k}}]$ for $t \in [j-1, j]$ to the j-th copy of S^1 in $\bigvee_{i=1}^k S^1$ by the map $t \mapsto e^{2\pi i(t-j+1)}$. The map g_k is just defined as the composition so that the left part of the diagram commutes. The map $\overline{f_k}$ is induced from f_k by the universal property of the quotient topology. For the collapse map $q_j\colon \bigvee_{i=1}^k S^1 \to S^1$ that maps all but the j-th copy of S^1 to the base point, we have $q_j \circ g_k \simeq \mathrm{id}_{S^1}$. Thus the composition of $\widetilde{H}_1(g_k)$ with the product map of $\widetilde{H}_1(q_1), \ldots, \widetilde{H}_1(q_k)$ is given by

$$(\widetilde{H}_1(q_1), \ldots, \widetilde{H}_1(q_k)) \circ \widetilde{H}_1(g_k) = (\mathrm{id}_{\widetilde{H}_1(S^1)}, \ldots, \mathrm{id}_{\widetilde{H}_1(S^1)})$$

Similarly, if $i_j\colon S^1 \to \bigvee_{i=1}^k S^1$ denotes the inclusion of the j-th copy of S^1, we have $(\bigvee_{i=1}^k \mathrm{id}_{S^1}) \circ i_j = \mathrm{id}_{S^1}$. Thus the composition of the coproduct map of $\widetilde{H}_1(i_1), \ldots, \widetilde{H}_1(i_k)$ with $\widetilde{H}_1(\bigvee_{i=1}^k \mathrm{id}_{S^1})$ is given by

$$\widetilde{H}_1(\textstyle\bigvee_{i=1}^k \mathrm{id}_{S^1}) \circ (\widetilde{H}_1(i_1) \oplus \cdots \oplus \widetilde{H}_1(i_k)) = \mathrm{id}_{\widetilde{H}_1(S^1)} \oplus \cdots \oplus \mathrm{id}_{\widetilde{H}_1(S^1)}$$

As we saw in (5.8), the product and coproduct maps appearing on the left of these equations are mutually inverse. Hence we can insert their composition in between the composition $\widetilde{H}_1(f_k) = \widetilde{H}_1(\bigvee_{i=1}^k \mathrm{id}_{S^1}) \circ \widetilde{H}_1(g_k)$ so that Example 1.33 yields

$$\begin{aligned}\widetilde{H}_1(f_k) &= (\mathrm{id}_{\widetilde{H}_1(S^1)} \oplus \cdots \oplus \mathrm{id}_{\widetilde{H}_1(S^1)}) \circ (\mathrm{id}_{\widetilde{H}_1(S^1)}, \ldots, \mathrm{id}_{\widetilde{H}_1(S^1)}) = \\ &= \mathrm{id}_{\widetilde{H}_1(S^1)} + \cdots + \mathrm{id}_{\widetilde{H}_1(S^1)} = k \cdot \mathrm{id}_{\widetilde{H}_1(S^1)}\end{aligned}$$

□

The $(n-1)$-fold suspension of f_k is a map $S^{n-1}(f_k)\colon S^n \to S^n$, which by naturality of the suspension isomorphism in Theorem 5.10 still has degree k. Hence maps of arbitrary degree exist in all dimensions. The point is that degree is a **complete invariant** of homotopy classes of self-maps of spheres: if $\deg f = \deg g$ for $f, g\colon S^n \to S^n$, then $f \simeq g$. The hard step in the proof is the following proposition.

Proposition 5.38
Let $f\colon S^n \to S^n$ *have* $\deg f = 0$. *Then* f *is null-homotopic.*

Proof We endow the n-sphere with a simplicial structure by fixing a homeomorphism $S^n \cong \partial\Delta^{n+1} =: K$. By simplicial approximation (Theorem 3.22), the map $f\colon K \to K$ is homotopic to a simplicial map on some iterated barycentric subdivision $K^{[r]}$ of the domain K. Since degree is homotopy invariant, we can thus assume $f\colon K^{[r]} \to K$ is simplicial to begin with.

Fix a vertex v in the codomain K. Since K is the boundary of a simplex, v has a unique opposing face τ. We consider a map $\Phi\colon (K, \overline{K\backslash\tau}) \longrightarrow (K, v)$, which stretches τ over the whole sphere K and maps the complement of the interior $\mathring{\tau}$ to v. Thus $\Phi \simeq \mathrm{id}_K$ and Φ restricts to a homeomorphism $\Phi|_{\mathring{\tau}} \xrightarrow{\cong} K\backslash\{v\}$. So we have $f \simeq g := \Phi \circ f$ and since f is simplicial, the map g sends each simplex σ in $K^{[r]}$ either to v or it maps $\partial\sigma$ to v and $\mathring{\sigma}$ homeomorphically to $K \setminus \{v\}$. In the former case, we call the simplex σ **ordinary**, in the latter case we call it **special**. Thus g factors as

$$g\colon K^{[r]} \longrightarrow \bigvee_{\sigma \text{ special}} S^n \xrightarrow{\bigvee_\sigma g_\sigma} K$$

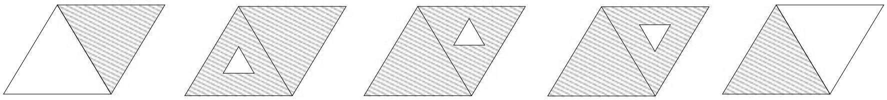


Fig. 5.1 By scaling, translating, rotating, and rescaling we can move a special simplex (white) to an adjacent ordinary simplex. The shaded area is mapped to the base point throughout

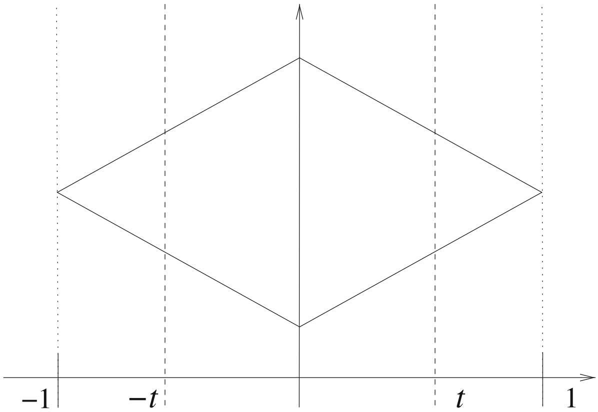


Fig. 5.2 The embedding $\sigma_- \cup \sigma_+ \subseteq \mathbb{R}^{n+1}$. The first axis represents $\mathbb{R}$, the second axis $\mathbb{R}^n$. The homotopy F_t equals $h(t, y) = h(-t, y)$ constantly along horizontal lines in the area between $-t$ and t. Whenever $(\pm t, y)$ lies outside $\sigma_- \cup \sigma_+$, the horizontal line thus maps to v so that in the end F_1 is the constant map to v

with pointed homeomorphisms $g_\sigma : (S^n, \bullet) \xrightarrow{\cong} (K, v)$ for each special simplex σ. As in the proof of Proposition 5.37, it follows that

$$\deg g = \sum_{\sigma \text{ special}} \deg(g_\sigma) \tag{5.39}$$

with $\deg(g_\sigma) = \pm 1$. The assumption $\deg = 0$ effects that there are as many "+1"s as "−1"s in this sum. By moves as indicated in Fig. 5.1 we can form "special pairs" (σ_-, σ_+) of simplices sharing precisely one face and satisfying $\deg g_{\sigma_-} = -1$ and $\deg g_{\sigma_+} = 1$. Since the maps $g_{\sigma_\pm}$ have opposite degree and come from simplicial homeomorphisms of n-simplices, we can scale, rotate, and rescale one of the two maps to achieve that the two maps are obtained from one another by reflection at the common face. We embed the union of the two simplices $\sigma_- \cup \sigma_+$ in $\mathbb{R}^{n+1}$ as indicated in Fig. 5.2 and extend $g|_{\sigma_- \cup \sigma_+}$ to $h \colon \mathbb{R}^{n+1} \to K$ by setting h constantly equal to v outside $\sigma_- \cup \sigma_+$. We then have $h(-x, y) = h(x, y)$ for all $(x, y) \in \mathbb{R}^{n+1}$. The homotopy

$$F(x, y, t) = \begin{Bmatrix} h(x, y), & |x| \geq t \\ h(t, y), & |x| \leq t \end{Bmatrix}$$

turns σ_- and σ_+ into ordinary simplices. By repeating this process we can cancel out all special simplices, which shows that g, thus f, is null-homotopic. $\square$

Remark 5.40
The formula (5.39) computes the "global degree" as a sum of "local degrees." As such, it has a differential topological version. Let $f: S^n \to S^n$ be smooth and let $q \in S^n$ be a regular value with $f^{-1}(q) = \{p_1, \ldots, p_k\}$. Then

$$\deg f = \sum_{l=1}^{k} \operatorname{sgn} \det \left(\frac{\partial f_i}{\partial x_j}(p_l) \right)$$

where the Jacobian $\frac{\partial f_i}{\partial x_j}(p_l)$ is computed with respect to charts around p_l and q, which are mapped to one another by rotations in the orientation preserving isometry group $\mathrm{SO}(n+1)$ of S^n. For a proof, see [4, Corollary 7.5, p. 192].

Now we are ready to prove that the notion of degree distinguishes all pointed homotopy classes of pointed self-maps of spheres.

Theorem 5.41
Let $n \geq 1$. Then the group $\pi_k(S^n, \bullet)$ is trivial for $1 \leq k < n$ and the map $\deg\colon \pi_n(S^n, \bullet) \xrightarrow{\cong} \mathbb{Z}$ is an isomorphism of groups.

Proof For $k < n$, the group $\pi_k(S^n, \bullet)$ is trivial because each map $S^k \to S^n$ is homotopic to a non-surjective one by simplicial approximation. To see that deg is a homomorphism, we recall from Sect. 2.5 that the product $[f] \cdot [g]$ of $[f], [g] \in \pi_n(S^n, \bullet)$, interpreted as homotopy classes of maps $(S^n, \bullet) \to (S^n, \bullet)$, is represented by

$$fg\colon S^n \longrightarrow S^n \vee S^n \xrightarrow{f \vee g} S^n$$

so that as before $\deg(fg) = \deg f + \deg g$. The kernel of deg is trivial by Proposition 5.38. To see surjectivity, either consider the reduced suspension $\Sigma^{n-1} f_k$ for $k \in \mathbb{Z}$ or, even simpler, just observe that $\deg(\mathrm{id}_{S^n}) = 1$ hits the generator. $\square$

For the passage back to unpointed homotopy, let us introduce the notation $[X, Y] = \mathrm{Hom}_{\mathsf{HoTop}}(X, Y)$ for unpointed homotopy classes of maps $f\colon X \to Y$.

Corollary 5.42
For $n \geq 1$, the map $\deg\colon [S^n, S^n] \xrightarrow{\cong} \mathbb{Z}$ is a bijection.

Proof Only for injectivity, we still need to argue that any given map $f: S^n \to S^n$ is homotopic to a map $\overline{f}$ with $\overline{f}(\bullet) = \bullet$. To do so, we observe that the rotation group $\mathrm{SO}(n+1)$ is path connected and acts transitively on S^n. Hence we can pick a path $r: I \to \mathrm{SO}(n+1)$ from the identity matrix to a rotation matrix that maps $f(\bullet)$ to $\bullet$. Then $r_t \circ f$ is a homotopy from f to a map $\overline{f}$ as required. □

Remark 5.43
More generally, the difference between pointed and unpointed homotopy classes of maps can be described as follows. For pointed spaces (X, x_0) and (Y, y_0), set $\langle X, Y\rangle = \mathrm{Hom}_{\mathsf{HoTop}_\bullet}((X, x_0), (Y, y_0))$. If (X, x_0) is well-pointed, we can define a right action of $\pi_1(Y, y_0)$ on $\langle X, Y\rangle$ by viewing a loop $\gamma: (I, \partial I) \to (Y, y_0)$ as a homotopy $\{x_0\} \times I \to Y$, which the HEP allows us to extend to a homotopy $H: X \times I \to Y$ starting at a given map $f: (X, x_0) \to (Y, y_0)$. So we can set $f \cdot [\gamma] = H_1$. If Y is path connected, the natural map $\langle X, Y\rangle \to [X, Y]$ descends to a bijection $\langle X, Y\rangle/\pi_1(Y, y_0) \cong [X, Y]$ on the orbit space, so $\langle X, Y\rangle \cong [X, Y]$ if Y is simply connected. For $X = S^1$, the action is just the conjugation action on the fundamental group, which is trivial if also $Y = S^1$ because $\pi_1(S^1, y_0)$ is abelian.

Proof of Theorem 5.33 Let $f: S^n \to S^n$ be any map and set $k := \deg f$. By Proposition 5.37 and the natural suspension isomorphism from $\widetilde{H}^{\mathrm{sing}}_*(S(-); \mathbb{Z})$ to $\widetilde{H}^{\mathrm{sing}}_{*-1}(-; \mathbb{Z})$, we have $\deg S^{n-1}(f_k) = k$. Hence Corollary 5.42 implies that $f \simeq S^{n-1}(f_k)$. Proposition 5.37 and the natural isomorphism from $\widetilde{H}_*(S(-))$ to $\widetilde{H}_{*-1}(-)$ also show that $\widetilde{H}_n(S^{n-1}(f_k)) = k \cdot \mathrm{id}_{\widetilde{H}_n(S^n)}$. Hence

$$\widetilde{H}_n(f) = \widetilde{H}_n(S^{n-1}(f_k)) = k \cdot \mathrm{id}_{\widetilde{H}_n(S^n)} = \deg f \cdot \mathrm{id}_{\widetilde{H}_n(S^n)}$$ □

5.5 Applications

Now that we have acquired a decent general knowledge on homology, the time is ripe to harvest some fruits of our work. In this section, we will work with the easiest homology theory we know: $H_*(-) = H^{\mathrm{sing}}_*(-; \mathbb{Z}/2)$ with values in $\mathbb{Z}/2$-vect.

The Fundamental Theorem of Algebra

The funny thing about the fundamental theorem of algebra is that it has no entirely algebraic proof. In fact, the complex numbers $\mathbb{C}$ are an analytic object, namely a degree two extension of the field of real numbers $\mathbb{R}$, which is constructed as a (metric or order theoretic) *completion* of $\mathbb{Q}$. This is why any proof of the fundamental theorem of algebra requires some analytic or topological input. Of course, as topologists, we prefer the latter.

Theorem 5.44
Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ be a polynomial with complex coefficients and no zeros in $\mathbb{C}$. Then $p(z)$ is constant.

Proof For all $t \in \mathbb{R}$, setting $f_t(z) = \frac{p(tz)}{|p(tz)|}$ defines a map $f_t : S^1 \to S^1$ because $p(z) \neq 0$ for all $z \in \mathbb{C}$. Since $H_s(z) = f_t((1-s)z)$ is a null-homotopy of f_t, we have $\deg(f_t) = 0$. Choose $r_0 > 0$ so that $|a_n z^n| > |a_{n-1} z^{n-1} + \cdots + a_0|$ for $|z| = r_0$. Consider the homotopy of polynomials $p_s(z) = a_n z^n + s(a_{n-1} z^{n-1} + \cdots + a_0)$ from which we obtain the homotopy of maps $g_s : S^1 \to S^1$ given by $g_s(z) = \frac{p_s(r_0 z)}{|p_s(r_0 z)|}$. Then $f_{r_0} = g_1 \simeq g_0$. Hence $0 = \deg f_{r_0} = \deg g_0 = \deg(z \mapsto z^n) = n$. □

Invariance of Dimension

It is conceivable that Euclidean spaces of different dimension should not be homeomorphic. But the existence of space filling curves is an indication that this might not be easy to prove. Yet homology is strong enough for this purpose.

Theorem 5.45
If $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$, then n is equal to m.

Proof A homeomorphism $f : \mathbb{R}^n \xrightarrow{\cong} \mathbb{R}^m$ induces a homotopy equivalence

$$S^{n-1} \xrightarrow{\simeq} \mathbb{R}^n \setminus \{0\} \xrightarrow{\cong} \mathbb{R}^m \setminus \{f(0)\} \xrightarrow{\simeq} S^{m-1}.$$

Thus $\widetilde{H}_{n-1}(S^{n-1}) \cong \widetilde{H}_{n-1}(S^{m-1})$ whence $n = m$ by Corollary 5.11. □

Nonexistence of Retractions

Here is another example where homology proves that a map with certain properties does not exist.

Theorem 5.46
The disk D^n does not retract onto its boundary S^{n-1}.

Proof Let $i\colon S^{n-1} \to D^n$ be the inclusion of the boundary. If $r\colon D^n \to S^{n-1}$ is a retraction so that $r \circ i = \mathrm{id}_{S^{n-1}}$, then $\mathrm{id}_{\widetilde{H}_{n-1}(S^{n-1})} = \widetilde{H}_{n-1}(r) \circ \widetilde{H}_{n-1}(i) = 0$ because $\widetilde{H}_{n-1}(D^n) = 0$ contradicting $\widetilde{H}_{n-1}(S^{n-1}) \cong \mathbb{Z}/2$. □

The Brouwer Fixed Point Theorem

The next classical theorem in topology is a neat example of an existence theorem proven by a nonexistence theorem. Fixed point theorems of this kind are relevant in game theory where they can be used to prove the existence of equilibria.

Theorem 5.47
Every map $f\colon D^n \to D^n$ has a fixed point.

Proof For $n = 0$, there is nothing to prove. Otherwise, the condition $f(x) \neq x$ for all $x \in D^n$ lets us define $r\colon D^n \to S^{n-1}$ by sending $x \in D^n$ to the intersection of the ray from $f(x)$ passing through x with S^{n-1}. Clearly r is continuous and restricts to the identity on S^{n-1}, which contradicts the last theorem. □

The Borsuk–Ulam Theorem

We now come to a more substantial application that unlike the preceding ones is based on the intrinsic mechanism of singular homology. So still let $H_*(-) = H_*^{\mathrm{sing}}(-;\mathbb{Z}/2)$ and for simplicity, we also use the notation $C_*(-) = C^{\mathrm{sing}}(-;\mathbb{Z}/2)$.

Theorem 5.48
For any map $f\colon S^n \to \mathbb{R}^n$, there is $x \in S^n$ with $f(x) = f(-x)$.

In addition to the calm points on Earth as a consequence of the hairy ball theorem, the Borsuk–Ulam theorem gives a continuation of our topological weather forecast. The case $n = 1$ implies that at each time and on each great circle on Earth, there exist two antipodal points with the same temperature. The case $n = 2$ implies that at each time, there exist two antipodal points on the surface of the Earth with the same temperature and the same atmospheric pressure. The theorem is an immediate consequence of the following result, known as the **antipode theorem**.

Theorem 5.49
Suppose the map $g\colon S^n \to S^m$ satisfies $g(-x) = -g(x)$ for all $x \in S^n$. Then $n \leq m$.

To see that the antipode theorem implies the Borsuk–Ulam theorem, assume that $f : S^n \to \mathbb{R}^n$ is a map with $f(x) \neq f(-x)$ for all $x \in S^n$. Then we can define a map $g : S^n \to S^{n-1}$ by setting

$$g(x) = \frac{f(x) - f(-x)}{\|f(x) - f(-x)\|}$$

so that $g(-x) = -g(x)$ for all $x \in S^n$. This contradicts the antipode theorem.

Proof of Theorem 5.49 The group with two elements $\langle t \mid t^2 \rangle \cong \mathbb{Z}/2$ acts on the k-sphere S^k by the antipodal map $t \cdot x = -x$. The orbit space can be identified with real projective k-space and the quotient map $S^k \xrightarrow{p} \mathbb{RP}^k$ is the 2-sheeted universal covering map. By covering theory, each singular p-simplex $\sigma : \Delta_p \to \mathbb{RP}^k$ has precisely two lifts $\bar{\sigma}_+, \bar{\sigma}_-$, which the generator t transforms into one another: we have $t\bar{\sigma}_\pm = \bar{\sigma}_\mp$. Correspondingly, we obtain a chain map

$$\tau_* : C_*(\mathbb{RP}^k) \longrightarrow C_*(S^k)$$

given by $\tau_*(\sigma) = \bar{\sigma}_+ + \bar{\sigma}_-$, which is called **transfer** and lies in the SES

$$0 \longrightarrow C_*(\mathbb{RP}^k) \xrightarrow{\tau_*} C_*(S^k) \xrightarrow{C_*(p)} C_*(\mathbb{RP}^k) \longrightarrow 0 \tag{5.50}$$

of singular chain complexes with coefficients in the field $\mathbb{Z}/2$. A map $g : S^n \to S^m$ with $g(-x) = -g(x)$ for all $x \in S^n$ descends to a map $\bar{g} : \mathbb{RP}^n \to \mathbb{RP}^m$ on the quotients. We thus obtain a "map" of SESes of chain complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_*(\mathbb{RP}^n) & \xrightarrow{\tau_*} & C_*(S^n) & \xrightarrow{C_*(p)} & C_*(\mathbb{RP}^n) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle C_*(\bar{g})} & & \downarrow{\scriptstyle C_*(g)} & & \downarrow{\scriptstyle C_*(\bar{g})} & & \\
0 & \longrightarrow & C_*(\mathbb{RP}^m) & \xrightarrow{\tau_*} & C_*(S^m) & \xrightarrow{C_*(p)} & C_*(\mathbb{RP}^m) & \longrightarrow & 0.
\end{array}$$

The right hand square commutes because the underlying square in Top commutes. To understand why the left hand square commutes, consider the diagram

$$\begin{array}{ccc}
\mathbb{RP}^n & \xleftarrow{p_n} & S^n \\
\downarrow{\scriptstyle \bar{g}} & & \downarrow{\scriptstyle g} \\
\mathbb{RP}^m & \xleftarrow{p_m} & S^m.
\end{array}$$

Let $\sigma : \Delta_p \to \mathbb{RP}^n$ and let $\bar{\sigma}_\pm$ be the unique two lifts of σ under p_n. Then

$$p_m \circ (g \circ \bar{\sigma}_\pm) = (p_m \circ g) \circ \bar{\sigma}_\pm = (\bar{g} \circ p_n) \circ \bar{\sigma}_\pm = \bar{g} \circ (p_n \circ \bar{\sigma}_\pm) = \bar{g} \circ \sigma$$

Thus $g \circ \sigma_\pm$ are the two unique lifts of $\bar{g} \circ \sigma$ under p_m.

By the naturality statement in Theorem 4.4 we thus obtain a commuting ladder of LESes. First we consider the lower LES of this ladder. It is clear that $\mathbb{RP}^m$ has an m-dimensional Δ-structure so that $H_{m+1}(\mathbb{RP}^m) = 0$ follows from Theorem 5.21. Thus the LES looks like

$$\begin{array}{ccccccc}
0 \longrightarrow & H_m(\mathbb{RP}^m) & \xrightarrow{H_m(\tau_*)} & H_m(S^m) & \xrightarrow{H_m(p)} & H_m(\mathbb{RP}^m) & \longrightarrow \\
\longrightarrow & H_{m-1}(\mathbb{RP}^m) & \longrightarrow & H_{m-1}(S^m) & \longrightarrow & H_m(\mathbb{RP}^m) & \longrightarrow \\
 & \cdots & & \cdots & \longrightarrow & H_1(\mathbb{RP}^m) & \longrightarrow \\
\longrightarrow & H_0(\mathbb{RP}^m) & \longrightarrow & H_0(S^m) & \longrightarrow & H_0(\mathbb{RP}^m). &
\end{array}$$

Note that in homology, the composition of covering map and transfer is trivial:

$$H_m(\tau_*) \circ H_m(p) = \mathrm{id}_{H_m(S^m)} + H_m(S^m \xrightarrow{\cdot t} S^m,\ x \mapsto -x) = 2 \cdot \mathrm{id}_{H_m(S^m)} = 0$$

because we are working over $\mathbb{Z}/2$. Since the LES says that $H_m(\tau_*)$ is injective, it thus follows that $H_m(p) = 0$ so the upper right most arrow in the L.E.S is zero and $H_m(\tau_*)$ is in fact an isomorphism. Since $H_k(S^m)$ vanishes for $0 < k < m$ by Corollary 5.11, the boundary homomorphisms $H_k(\mathbb{RP}^m) \longrightarrow H_{k-1}(\mathbb{RP}^m)$ are isomorphisms at least whenever $1 < k \leq m$. But the final arrow of the LES is clearly an isomorphism because both S^m and $\mathbb{RP}^m$ are path connected. So the preceding arrow is zero and hence also $H_1(\mathbb{RP}^m) \longrightarrow H_0(\mathbb{RP}^m)$ is an isomorphism. Since the same remarks apply for the upper LES, the square with the boundary homomorphism of the ladder in case $n \geq m$ and $1 \leq i \leq m$ looks like

$$\begin{array}{ccc}
H_i(\mathbb{RP}^n) & \xrightarrow[\cong]{\partial} & H_{i-1}(\mathbb{RP}^n) \\
{\scriptstyle H_i(\bar{g})}\downarrow & & \downarrow{\scriptstyle H_{i-1}(\bar{g})} \\
H_i(\mathbb{RP}^m) & \xrightarrow[\cong]{\partial} & H_{i-1}(\mathbb{RP}^m).
\end{array}$$

Since $H_0(\bar{g})$ is an isomorphism, the diagram says that so is $H_i(\bar{g})$ for $1 \leq i \leq m$ by induction. The last one of these, $H_m(\bar{g})$, sits also in the square

$$\begin{array}{ccc}
H_m(\mathbb{RP}^n) & \xrightarrow{H_m(\tau_*)} & H_m(S^n) \\
{\scriptstyle H_m(\bar{g})}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle H_m(g)} \\
H_m(\mathbb{RP}^m) & \xrightarrow[\cong]{H_m(\tau_*)} & H_m(S^m).
\end{array}$$

of the ladder. But for $n > m$, this is absurd because from Corollary 5.11, we have $H_m(S^n) = 0$ while $H_m(S^m) \cong \mathbb{Z}/2$. □

Karol Borsuk (1905 in Warsaw—1982 at the same place) was a Polish mathematician known, among other achievements, to be the founder of shape theory, a version of homotopy theory better behaved for pathological spaces, like the Warsaw circle that he likewise discovered. One of his students was Samuel Eilenberg who jointly with Saunders Mac Lane founded category theory. During the German occupation of Warsaw, Borsuk developed the dice game "Animal Husbandry" in 1943 and sold copies of them to support his family [11]. It seemed that most of these copies were lost during the Warsaw Uprising but in the 1990s an intact set was found and the game is now available on the market under different names. Notably, the game uses dodecahedra as 12-sided dice.

Stanisław Marcin Ulam (1909 Lwów, Poland, now Lviv, Ukraine—1984 Santa Fe, New Mexico) was a Polish–American mathematician with a particularly wide range of research interests. In his early years, he was part of the "Lwów school of mathematics" known for studying fundamental problems in point-set topology, set theory, and functional analysis during long hours in the Scottish cafe in Lwów. Later he emigrated to the USA where in 1944 he joined the secret Manhattan project in Los Alamos, New Mexico, which developed the first nuclear weapons [30]. Modern thermonuclear weapons still employ the Teller–Ulam design, which Ulam developed post World War–II in 1951 with Hungarian–American physicist Edward Teller. Ulam is also known for discovering the Ulam spiral: Arraying the integers along a spiral, the prime numbers astonishingly often occur along horizontal, diagonal, and vertical lines. Such lines correspond to quadratic functions some of which were known to oftentimes produce primes already by Euler, but the phenomenon still lacks an encompassing explanation. Another outcome of the Manhattan project due to Ulam is the Monte Carlo method of sampling with pseudo random numbers to find numerical solutions for problems that would otherwise require unfeasible calculations. The idea has proven to be fruitful in a variety of areas, and is for example still applied to derivative pricing in mathematical finance.

The Ham Sandwich Theorem

The Borsuk–Ulam theorem has the following beautiful corollary. It asserts that a ham sandwich consisting of two layers of bread and one layer of ham—no matter how they are shaped, no matter how they are positioned—can be cut with a single straight slice of a knife such that each layer is cut into halves of the same size.

Theorem 5.51
Let $A_1, \dots, A_m \subseteq \mathbb{R}^m$ be Borel sets of finite Lebesgue measure $\lambda(A_i) < \infty$ for all i. Then there exists an affine hyperplane in $\mathbb{R}^m$, which dissects each A_i into subsets of equal measure.

Proof We embed $\mathbb{R}^m$ affinely as $\mathbb{R}^m \times \{1\} \subseteq \mathbb{R}^{m+1}$. Given $x \in \mathbb{R}^{m+1} \setminus \{0\}$, set

- $H_x^+ = \{y \in \mathbb{R}^{m+1} : \langle x, y\rangle \geq 0\} \cap \mathbb{R}^m \times \{1\}$,
- $H_x^- = \{y \in \mathbb{R}^{m+1} : \langle x, y\rangle \leq 0\} \cap \mathbb{R}^m \times \{1\}$,
- $H_x = H_x^+ \cap H_x^-$.

Let $f_i : S^m \to \mathbb{R}$ be given by $f_i(x) = \lambda(A_i \cap H_x^+)$. Then it is apparent that $f_i(-x) = \lambda(A_i \cap H_x^-)$. Considering the product map $f : S^m \to \mathbb{R}^m$ given by $f(x) = (f_1(x), \dots, f_m(x))$, the theorem follows from the Borsuk–Ulam theorem once we know that f is continuous. Given $x \in S^m$ and a sequence (x_n) of points in S^m with $\lim_{n\to\infty} x_n = x$, the characteristic functions $\chi_{A_i \cap H_{x_n}^\pm}$ converge pointwise to $\chi_{A_i \cap H_x^\pm}$, except possibly on $A_i \cap H_x$. Thus they converge λ-almost everywhere and they are bounded from above by the integrable function χ_{A_i}. The dominated convergence theorem implies

$$\lim_{n\to\infty} f_i(x_n) = \lim_{n\to\infty} \int \chi_{A_i \cap H_{x_n}^+} \, d\lambda = \int \chi_{A_i \cap H_x^+} \, d\lambda = f_i(x)$$

□

Exercises

5.1 Find a pair of spaces (X, A) such that $H_1^{\text{sing}}(X, A; \mathbb{Z})$ is not isomorphic to the reduced homology group $\widetilde{H}_1^{\text{sing}}(X/A; \mathbb{Z})$.

5.2 Let (X, x_0) be a well-pointed space. Show that the arc inclusion $x_0 \times I \subset SX$ is a cofibration. *Hint: You may assume $\{x_0\} \subseteq X$ is closed. Find an NDR presentation of $X \times \{0, 1\} \cup \{x_0\} \times I \subset X \times I$ and show that it descends to an NDR presentation of $x_0 \times I \subset SX$. The general case works similarly with Strøm's characterization of cofibrations in [25, Lemma 4].*

5.3 Let (H_*, ∂_*) be a homology theory with values in R-mod satisfying the dimension axiom and $H_0(\bullet) \neq 0$. Let $A \subset S^n$ be a *proper* subset. Show that $H_n(S^n, A)$ is not trivial.

5.4 Let (H_*, ∂_*) be a homology theory with values in R-mod satisfying the dimension axiom.

(a) Apply the Mayer–Vietoris sequence for pushouts to compute $H_k(S^n)$.
(b) Using the result for $H_k(S^1)$, find the homology of the 2-torus $H_k(\mathbb{T}^2)$ again as an application of the Mayer–Vietoris sequence for pushouts.

5.5 Let (H_*, ∂_*) be a homology theory with values in R-mod. Show that we have a natural isomorphism $H_k(X \times S^n) \cong H_k(X) \oplus H_{k-n}(X)$.

Hint: Let $x_0 \in S^n$ and show first that

$$H_k(X \times S^n) \cong H_k(X) \oplus H_k(X \times S^n, X \times \{x_0\}).$$

Afterwards, show that $H_k(X \times S^n, X \times \{x_0\}) \cong H_{k-n}(X)$ with the help of the Mayer–Vietoris sequence. You can either find a suitable excisive triad or you may assume that X is locally compact and find a suitable pushout.

5.6 Suppose (H_*, ∂_*) satisfies the dimension axiom. Show that

$$H_k(\mathbb{T}^n) \cong H_0(\bullet)^{\binom{n}{k}}.$$

5.7 For each $n \geq 1$ find a surjective map $S^n \xrightarrow{f} S^n$ of degree zero.

5.8 Let $f_k \colon S^1 \to S^1$, $z \mapsto z^k$. Show that $\deg(f_k) = k$ also follows from naturality of the Hurewicz isomorphism.

5.9 Show that for every map $f \colon S^{2n} \to S^{2n}$, there is $x \in S^{2n}$ with $f(x) \in \{x, -x\}$. Conclude that every map $\mathbb{RP}^{2n} \to \mathbb{RP}^{2n}$ has a fixed point. How about maps $\mathbb{RP}^{2n+1} \to \mathbb{RP}^{2n+1}$?

5.10 Show that homeomorphic manifolds have equal dimension.

Cellular Homology

6

In Example 5.30, we saw that the Mayer–Vietoris sequence gives us good control on the effect on homology of attaching a cell to a space. Say a space X arises entirely from attaching cells. This could mean that a sequence of nested subspaces $X^0 \subseteq X^1 \subseteq \cdots \subseteq X$ forms an exhaustion $X = \bigcup_n X^n$ such that X^0 is discrete and such that X^n arises inductively from X^{n-1} by attaching n-cells along attaching maps $f_i^n \colon S^{n-1} \to X^{n-1}$. By collapsing the complement of the interior of the j-th attached $(n-1)$-cell in X^{n-1}, we obtain a quotient map $p_j \colon X^{n-1} \to S^{n-1}$. The quotient maps corresponding to the $(n-1)$-cells decompose the attaching maps f_i^n of the n-cells into families of maps $f_{ij}^n = p_j \circ f_i \colon S^{n-1} \to S^{n-1}$. The Mayer–Vietoris sequence makes it now conceivable that at least for an ordinary homology theory (H_*, ∂_*), the induced morphisms $\widetilde{H}_{n-1}(f_{ij}^n)$ for ranging n, i, and j contain the complete information on the homology of X. By Theorem 5.33, this information is in turn encoded in the integers $\deg f_{ij}^n$. So two remarkable consequences come to mind. Firstly, there should be a combinatorial way of computing ordinary homology from the numbers $\deg f_{ij}^n$. Secondly, these numbers are independent of (H_*, ∂_*), so ordinary homology should be determined on **cell complexes**, or more precisely on **CW complexes**, like our space X, by the Eilenberg–Steenrod axioms. Since many spaces, including Δ-complexes and smooth manifolds, admit CW structures, one could regard this fact as the ultimate justification of the axioms.

6.1 CW Complexes

To make the above ideas rigorous, we start with a formal introduction of the spaces of interest in this chapter.

Definition 6.1

A **CW complex** consists of a topological space X and a filtration

$$\emptyset = X^{-1} \subseteq X^0 \subseteq X^1 \subseteq \cdots \subseteq X$$

H. Kammeyer, *Introduction to Algebraic Topology*, Compact Textbooks in Mathematics, https://doi.org/10.1007/978-3-030-98313-0_6

by subspaces such that for each $n \geq 0$, there exists a family of maps $Q_i^n \colon D^n \to X^n$ for $i \in I_n$ that restrict to maps $q_i^n \colon S^{n-1} \to X^n$ such that:

(i) The square

$$\begin{array}{ccc} \coprod_{i \in I_n} S^{n-1} & \xrightarrow{\coprod q_i^n} & X^{n-1} \\ \downarrow & & \downarrow j_n \\ \coprod_{i \in I_n} D^n & \xrightarrow{\coprod Q_i^n} & X^n \end{array} \tag{6.2}$$

is a pushout in Top.

(ii) We have $X = \operatorname{colim}_n X^n$ in Top.

To verify that the definition does what we want it to do, let us decode it step by step. For $n = 0$, condition (i) says that

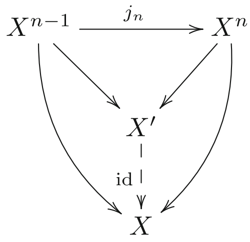
$$\begin{array}{ccc} \emptyset & \longrightarrow & \emptyset \\ \downarrow & & \downarrow \\ \coprod_{i \in I_0} \{\bullet\} & \xrightarrow{\cong} & X^0 \end{array}$$

is a pushout, which shows that the subspace X^0 carries the discrete topology. For $n \geq 1$, the condition says that the ***n*-skeleton** X^n is obtained from X^{n-1} by attaching copies of the n-disk D^n along attaching maps $q_i^n \colon S^{n-1} \to X^{n-1}$ indexed by the index set I_n. Theorem 1.42 (ii) implies that the inclusions $j_n \colon X^{n-1} \to X^n$ are closed and that the path components of $X^n \setminus X^{n-1}$ are homeomorphic to open n-disks. We call the path component $e_i^n = Q_i^n(D^n \setminus S^{n-1})$ the i-th **open *n*-cell** of X, and we refer to Q_i^n as the **characteristic map** of the cell e_i^n.

Condition (ii) says that we have $X = \bigcup_{n \geq -1} X^n$ as a set and that X is **coherent** with the skeleta X^n, meaning X carries the **final topology** with respect to the inclusions $i_n \colon X^n \to X$. Explicitly, a subset $D \subseteq X$ is closed (or open) if and only if $D \cap X^n \subseteq X^n$ is closed (or open) for all n. On the one hand, this is the maximal choice of closed subsets, such that X is a cocone on the diagram of inclusions j_n in Top,

$$\begin{array}{ccc} X^{n-1} & \xrightarrow{j_n} & X^n \\ & {\scriptstyle i_{n-1}} \searrow \quad \swarrow {\scriptstyle i_n} & \\ & X & \end{array},$$

meaning all inclusions $i_n\colon X^n \to X$ are still continuous. On the other hand, universality requires that X have no less closed sets. If X' equals X as a set but has less closed sets as topological space, then $\mathrm{id}\colon X' \to X$ is no longer continuous, violating the universal property, which requires that this arrow uniquely completes the diagram



as a diagram in Top. Since the inclusions $j_n\colon X^{n-1} \to X^n$ are closed, so are the inclusions $i_n\colon X^n \to X$.

We stress that while the filtration $(X^n)_{n=-1}^{\infty}$ is part of the structure of a CW complex, the characteristic maps, or equivalently the pushouts in (i), are *not*. Only their existence is required. In more detail, this still means that the families of open n-cells $\{e_i^n\}_{i\in I_n}$ are part of the structure as they are the path components of $X^n \setminus X^{n-1}$. Hence we can describe the index sets as $I_n = \pi_0(X^n \setminus X^{n-1})$. But the characteristic maps Q_i^n and their restrictions to gluing maps q_i^n are only available after an arbitrary choice. Accordingly, the notion of morphisms of CW complexes and the construction of functors from the category of CW complexes must be independent of any such choice. Let us check that CW complexes have a good point-set topology.

Theorem 6.3
CW complexes are T_4 spaces (normal and Hausdorff).

Proof Recall that a space is called normal if any two disjoint closed subsets have disjoint open neighborhoods. Hence it is enough to show that a CW complex X is normal and T_1, meaning points are closed. To see the latter, let $x_0 \in X$ and let m be minimal with $x_0 \in X^m$. Choosing an attaching pushout, we obtain a quotient map $q_m\colon (\coprod_{i\in I_m} D^m) \coprod X^{m-1} \longrightarrow X^m$ and $q_m^{-1}(x_0)$ is a single interior point in some copy of D^m, hence closed. Thus x_0 is closed in X^m by Lemma A.1, hence closed in X^n for $n \geq m$ because the maps $j_n\colon X^{n-1} \to X^n$ are closed inclusions. By the description of the final topology on X above, x_0 is closed in X.

To see that X is normal, we first show by induction that each skeleton X^n is normal. The empty space X^{-1} is normal because $\emptyset, \emptyset \subseteq X^{-1}$ are separated by $\emptyset, \emptyset \subseteq X^{-1}$. In view of condition (i) in the definition of CW complexes, it is now enough to see that for a pushout

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ B & \xrightarrow{\overline{f}} & Z \end{array}$$

in Top with a closed embedding i, we have that Z is normal if A, B, and C are normal. To prove this, we use the Tietze extension characterization of normality. So let $D \subseteq Z$ be closed, and let $g\colon D \to \mathbb{R}$ be continuous. Then we have closed subsets $D_B = \overline{f}^{-1}(D)$ and $D_C = j^{-1}(D)$ of B and C, respectively. The restriction of $g \circ j$ to D_C extends to a map $g_C\colon C \to \mathbb{R}$ because C is normal. The set $i(A) \cup D_B$ is closed in B and carries the pushout topology of $i(A) \leftarrow i(A) \cap D_B \to D_B$. Since i is an embedding, sending $i(a) \in i(A)$ to $g_C(f(a))$ and $x \in D_B$ to $g(\overline{f}(x))$ gives a well-defined and continuous map $i(A) \cup D_B \to \mathbb{R}$, which by normality of B extends to $g_B\colon B \to \mathbb{R}$. Finally, g_B and g_C form a cocone on the above pushout, so they define a map $g_Z\colon Z \to \mathbb{R}$ that extends g.

Now let $A \subseteq X = \operatorname{colim}_n X^n$ be closed, and let $g\colon A \to \mathbb{R}$ be continuous. We obtain closed subsets $A_n := A \cap X^n$ of the normal space X^n, and, arguing inductively as above, we find compatible extensions $g_n\colon X^n \to \mathbb{R}$ of $g|_{A_n}$ so that $g_n|_{X^{n-1}} = g_{n-1}$. Hence the g_n form a cocone on $X^{-1} \to X^0 \to X^1 \to X^2 \to \cdots$, and we get a unique map $g_\infty\colon X \to \mathbb{R}$ extending g as required. □

Since compact subsets of Hausdorff spaces are closed, we conclude that the **closed n-cells** $\overline{e}_i^n = Q_i^n(D^n)$ are indeed the closures of the open cells e_i^n in X.

Theorem 6.4
Let X be a CW complex, and let $C \subseteq X$ be any subset.

(i) The set C is closed if and only if $C \cap \overline{e}_i^n$ is compact for each closed cell $\overline{e}_i^n$.
(ii) The set C is compact if and only if C is closed and meets only finitely many open cells e_i^n.

Proof If $C \subseteq X$ is closed, then $C \cap \overline{e}_i^n$ is a closed subset of the compact set $\overline{e}_i^n$, hence is compact. So let conversely $C \cap \overline{e}_i^n$ be compact for each closed cell $\overline{e}_i^n$. Trivially, $C \cap X^{-1} = \emptyset$ is closed. For given $n \geq 0$, we choose a pushout as in (6.2). By Lemma 1.35, the map $(\coprod Q_i^n) \coprod j_n\colon (\coprod_{i \in I_n} D^n) \coprod X^{n-1} \longrightarrow X^n$ is an identification map. To see that $C \cap X^n$ is closed, it is by Lemma A.1 enough to show that $j_n^{-1}(C \cap X^n) = C \cap X^{n-1}$ is closed, which is true by induction hypothesis, and that $(\coprod Q_i^n)^{-1}(C \cap X^n)$ is closed, which is true because $C \cap Q_i^n(D^n) = C \cap \overline{e}_i^n$ is closed by assumption, being a compact subset of a Hausdorff space. So $C \cap X^n \subseteq X^n$ is closed for all n, hence $C \subseteq X$ is closed, which proves (i).

If C is closed and lies in the union of finitely many open cells, it also lies in the union of the closures of these cells and hence is a closed subset of a compact space, hence C is compact.

Conversely, let $C \subseteq X$ be compact. Then C is closed because X is Hausdorff by Theorem 6.3. Suppose that C was not contained in any skeleton of X. Then we can find an infinite subset $D \subseteq C$ consisting of points lying in open cells of pairwise different dimension. Then every subset of D has finite intersection with every skeleton X^n. In particular, every subset of D is closed. This shows that the subset $D \subseteq C$ is infinite, closed, and discrete, contradicting that C is compact. Hence we must have $C \subseteq X^n$ for some n. Suppose that C met infinitely many open cells. Then there is $k \leq n$ and an infinite subset $D \subseteq C$ such that each point in D lies in a different open k-cell. Then for $l \leq k$, the intersection of every subset of D with each closed l-cell either is empty or consists of a single point, hence is compact. By (i), this shows that D is an infinite, discrete, and closed subset of the compact space $C \cap X^k$, which is absurd. This proves (ii). □

The theorem explains the term "CW complex." The "C" refers to "closure-finite," meaning each closed cell $\overline{e}_i^n$ meets only a finite number of other cells. The "W" refers to "weak topology" because a CW complex carries a weak topology in the sense that one can already inspect on skeleta whether a given subset is open or closed. From the theorem, it is immediate that a subset $C \subseteq X$ is compact if and only if C is closed and lies in a finite subcomplex. In particular, a CW complex X is compact if and only if it is finite. To have a pile of examples of CW complexes, let us endow those manifolds that are typically of interest in algebraic topology with explicit CW structures.

Example 6.5 The ***n*-sphere** $X = S^n$ has a CW structure with only one 0-cell and one n-cell. So the filtration is $X^{-1} = \emptyset$, $X^0 = X^1 = X^2 = \cdots = X^{n-1} = \{\bullet\}$, and $X^n = S^n$ and there is only one possible choice of pushouts given by:

$$\begin{array}{ccc} \emptyset & \longrightarrow & X^{k-1} \\ \downarrow & & \downarrow \\ \emptyset & \longrightarrow & X^k \end{array}$$

for $k = 1, \ldots, n-1$ and by:

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & X^{n-1} = \{\bullet\} \\ \downarrow & & \downarrow \\ D^n & \longrightarrow & X^n = S^n \end{array}$$

for $k = n$.

Example 6.6 The **orientable surface** $X = \Sigma_g$ of genus g has a CW structure with one 0-cell, $2g$ many 1-cells, and one 2-cell. The filtration is given by $X^{-1} = \emptyset$, $X^0 = \{\bullet\}$, $X^1 = \bigvee_{2g} S^1$, and $X^2 = X$. The only pushout of interest is the one attaching the 2-cell

$$\begin{array}{ccc} S^1 & \longrightarrow & \bigvee_{2g} S^1 \\ \downarrow & & \downarrow \\ D^2 & \longrightarrow & X \end{array}$$

which was described in Example 2.29.

Example 6.7 Real projective d-space $X = \mathbb{RP}^d$ has a CW structure with one cell in each degree from zero to d. The filtration can be given in homogeneous coordinates as $X^n = \{[x_0 : \cdots : x_n : 0 : \cdots : 0] \in \mathbb{RP}^d\} \cong \mathbb{RP}^n$ and the cells are attached by:

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{q^n} & X^{n-1} \cong \mathbb{RP}^{n-1} \\ \downarrow & & \downarrow \\ D^n & \xrightarrow{Q^n} & X^n \cong \mathbb{RP}^n \end{array}$$

with $q_n(x_1, \ldots, x_n) = [x_1 : \cdots : x_n]$, which is the 2-fold covering map and $Q^n(x_1, \ldots, x_n) = [x_1 : \cdots : x_n, \sqrt{1 - \|(x_1, \ldots, x_n)\|^2}]$, which is the orthogonal projection from the n-disk to the upper hemisphere of the n-sphere followed by the 2-fold covering map.

Example 6.8 Complex projective *d*-space $X = \mathbb{CP}^d$ has one $2n$-cell in each even degree $0 \leq 2n \leq 2d$. The filtration can again be described in homogeneous coordinates as $X^{2n} = X^{2n+1} = \{[z_1, \ldots, z_n, 0, \ldots, 0] \in \mathbb{CP}^d\} \cong \mathbb{CP}^n$. The cells are attached by

$$\begin{array}{ccc} S^{2n-1} & \xrightarrow{q^{2n}} & X^{2n-1} \cong \mathbb{CP}^{n-1} \\ \downarrow & & \downarrow \\ D^{2n} & \xrightarrow{Q^{2n}} & X^{2n} \cong \mathbb{CP}^n \end{array}$$

As above, gluing and attaching maps are given by:

$$q^{2n}(z_1, \ldots, z_n) = [z_1 : \cdots : z_n]$$
$$Q^{2n}(z_1, \ldots, z_n) = [z_1 : \cdots, z_n : \sqrt{1 - \|(z_1, \ldots, z_n)\|^2}].$$

Remark 6.9

The gluing map $S^3 \xrightarrow{q^4} \mathbb{CP}^1 \cong S^2$ of the 4-cell in the above CW structure of the complex projective plane $\mathbb{CP}^2$ is the famous **Hopf bundle** generating $\pi_3(S^2, \bullet) = \langle [q^4] \rangle \cong \mathbb{Z}$. A visualization of this map has made it to the title page of Hatcher's book [8].

It is convenient to have the notion of a *relative CW complex* (X, A) available, which should be a space X that arises from a given space A by attaching cells. In view of Theorem 6.3, we would like to preserve at least the Hausdorff property for this concept so that we define a **relative CW complex** (X, A) by allowing $X^{-1} = A$ instead of only $X^{-1} = \emptyset$ in the definition of CW complexes but we require that the space A be Hausdorff. The categories CW and relCW of CW complexes and relative CW complexes are defined by the following notion of morphisms.

Definition 6.10
A map $f: (X, A) \to (Y, B)$ of relative CW complexes is **cellular** if $f(X^n) \subseteq Y^n$ for all $n \geq -1$.

Also of interest is the notion of a **CW pair** (X, A) consisting of an (absolute) CW complex X and a **subcomplex** $A \subseteq X$: a closed union of open cells.

Lemma 6.11
A subcomplex $A \subseteq X$ of a CW complex is a CW complex.

Proof We claim that $A^n := A \cap X^n$ is a filtration by closed subsets that defines a CW structure on A. Choose pushouts for X, and let $I_n(A) \subseteq I_n$ be the subset indexing the n-cells in A. Then $(\coprod_{i \in I_n(A)} Q_i^n) \coprod (j_n|_{A \cap X^{n-1}})$ is a quotient map because $(\coprod_{i \in I_n} Q_i^n) \coprod j_n$ is a quotient map and because A is closed. This gives Condition (i). Condition (ii) follows because A being closed implies $C \subseteq A \cap X^n$ is closed if and only if C is closed in X^n. □

Morphisms $f: (X, A) \to (Y, B)$ of CW pairs are cellular maps $f: X \to Y$ with $f(A) \subseteq B$. Thus f restricts to a cellular map $f|_A: A \to B$. We will denote the category of CW pairs by $\mathsf{CW}^{(2)}$. Since subcomplexes are CW complexes by the lemma and CW complexes are Hausdorff by Theorem 6.3, we have a functor $\mathsf{CW}^{(2)} \to \mathsf{relCW}$, which sends the filtration $(X^n)_{n=-1}^{\infty}$ to $(X^n \cup A)_{n=-1}^{\infty}$. In this sense, we may consider a CW pair as a special type of relative CW complex.

Definition 6.12
Let (X, A) be a relative CW complex. We say that (X, A) has **dimension** n and write $\dim(X, A) = n$ if $X = X^n$ and $X \neq X^{n-1}$. We say (X, A) has **finite type** if it has finitely many cells in each degree. We say that (X, A) is **finite**, if it has finitely many cells altogether.

Given a CW pair (X, A) with $A \neq \emptyset$, it is sometimes useful to use the term **relatively finite** and the like when the definition is applied to (X, A) and **absolutely finite** when it is applied to $(X, \emptyset)$. Note that finite implies finite dimensional but not conversely. If (X, A) is finite dimensional, it might not even have finite type. Let us briefly explain how to obtain new CW complexes out of old. We observe that if X and Y are (relative) CW complexes, then clearly so is the coproduct $X \coprod Y$. For the product, however, we need a point-set topological condition.

Theorem 6.13
Suppose that one of the CW complexes X and Y is locally compact. Then the product space $X \times Y$ is a CW complex with filtration

$$(X \times Y)^n = \bigcup_{p+q=n} X^p \times Y^q .$$

Proof Every open n-cell in $X \times Y$ is the product $e_p^{X,n} \times e_q^{Y,n}$ of an open p-cell in X and an open q-cell in Y with $p + q = n$. One finds a corresponding characteristic map using a homeomorphism $D^n \cong D^p \times D^q$. The subtlety in need of attention concerns the topology. Unifying all characteristic maps of X and Y to maps $\coprod_{n\geq 0} \coprod_{i\in I_n^X} Q_i^{X,n}$ and $\coprod_{n\geq 0} \coprod_{i\in I_n^Y} Q_i^{Y,n}$, these maps are by assumption identification maps

$$(\textstyle\coprod_{n\geq 0} \coprod_{i\in I_n^X} D^n) \longrightarrow X, \qquad (\textstyle\coprod_{n\geq 0} \coprod_{i\in I_n^Y} D^n) \longrightarrow Y$$

and we have to show that so is their product. Without loss of generality, we may assume X is locally compact. Since the product map can be factored as

$$\left(\coprod_{n,i} Q_i^{X,n}\right) \times \left(\coprod_{n,i} Q_i^{Y,n}\right) = \left(\coprod_{n,i} Q_i^{X,n} \times \mathrm{id}_{\coprod_{n,i} D^n}\right) \circ \left(\mathrm{id}_X \times \coprod_{n,i} Q_i^{Y,n}\right),$$

it is an identification map because it is the composition of two identification maps according to Proposition A.2 (ii). □

Note that the above filtration $(X \times Y)^n$ also in general defines a CW structure on $X \times Y$, but the corresponding coherent topology might be finer (have more open sets) than the product topology of $X \times Y$.

If X is a CW complex and $A \subseteq X$ is a subcomplex, then X/A and hence the cone CX, the suspension SX, and the reduced suspension ΣX are CW complexes. If $Y \xrightarrow{p} X$ is a covering of a CW complex X, then $Y^n = p^{-1}(X^n)$ defines the filtration of a CW structure on Y (Exercises 6.1, 6.2 and 6.3). The key reason why CW complexes are homotopy theoretically well-behaved is the following theorem.

Theorem 6.14
A relative CW complex (X, A) is a closed cofibration.

Proof The argument from below Definition 6.1 showing that the inclusions of skeleta $i_n : X^n \to X$ are closed still applies to relative CW complexes so that $X^{-1} = A \subseteq X$ is closed. Applying Theorem 2.21 to a choice of cell attaching pushouts as in Definition 6.1 (i), we see that the inclusions $j : X^{n-1} \to X^n$ are cofibrations. Since $X^{-1} = A$ and since trivially the composition of cofibrations are cofibrations, we obtain that (X^n, A) is a cofibration for all $n \geq -1$.

Let $i : A \to X$ be the inclusion, let $H : A \times I \to Y$ be a homotopy, and let $f : X \to Y$ be an initial condition so that $f \circ i = H_0$. Then inductively, the HEP of the cofibration (X^n, A) applied to the initial condition $f \circ i_n$ gives an extension $H'^n : X^n \times I \to Y$ of the homotopy $H'^{n-1} : X^{n-1} \times I \to Y$ with $H'^{-1} = H$. Since I is compact, we have $\mathrm{colim}_n(X^n \times I) = X \times I$ by Proposition 1.44. So we get a unique map $H' : X \times I \to Y$ extending all the maps H'^n for $n \geq -1$. In particular H' restricts to H on $A \times I$ and $H'_0 = f$. This shows the HEP of i for any Y. □

Consequently, the LES of Theorem 5.7 applies to nonempty relative CW complexes and in particular to nonempty CW pairs. Moreover:

Corollary 6.15
Let X be a nonempty CW complex, and let $x_0 \in X^0$ be a 0-cell. Then (X, x_0) is well-pointed.

Combining the theorem with Corollary 2.25 gives the following.

Corollary 6.16
Let X be a CW complex, and let $A \subseteq X$ be a contractible subcomplex. Then X is homotopy equivalent to X/A.

Here is a convenient application of this fact.

Corollary 6.17
Let X be a path connected CW complex. Then X is homotopy equivalent to a CW complex with a single 0-cell.

Proof As the attaching maps of n-cells for $n \geq 2$ have path connected image, already the 1-skeleton X^1 must be path connected. The 1-skeleton X^1 is a graph, which by Zorn's lemma has a maximal subtree $T \subseteq X^1$. Since T is a contractible subcomplex of X, the collapse space X/T is a CW complex, which does the job. □

This shows that the assumption in the following description of the fundamental group of a path connected CW complex comes with no loss of generality.

Theorem 6.18
Let X be a path connected CW complex with a single 0-cell x_0. Choose characteristic maps $(Q_i^n, q_i^n)\colon (D^n, S^{n-1}) \longrightarrow (X^n, X^{n-1})$ for $i \in I_n$ and $n = 1, 2$. Then $\pi_1(X, x_0) \cong \left\langle [Q_i^1],\ i \in I_1 \,\middle|\, [q_j^2],\ j \in I_2 \right\rangle$.

In this presentation of the group $\pi_1(X, x_0)$, it is understood that Q_i^1 and q_j^2 are precomposed with the maps $I \to D^1, t \mapsto 2t - 1$ and $I \to S^1, t \mapsto \exp(2\pi i t)$, respectively, so that they define generators and relators in the free group $\pi_1(X^1, x_0)$.

Proof The homomorphisms $\pi_1(i_n)\colon \pi_1(X^n, x_0) \to \pi_1(X, x_0)$ form a cocone on the diagram $\pi_1(X^1, x_0) \to \pi_1(X^2, x_0) \to \cdots$, and we claim that the corresponding unique homomorphism $\operatorname{colim}_n \pi_1(X^n, x_0) \to \pi_1(X, x_0)$ is an isomorphism. To see it is surjective, let $[\gamma] \in \pi_1(X, x_0)$. Then $\gamma(I) \subseteq X$ is compact, hence lies in a finite subcomplex by Theorem 6.4 (ii), so $\gamma(I) \subseteq X^n$ for a large enough n, hence $[\gamma] \in \operatorname{im} \pi_1(i_n)$. To see injectivity, let $[\gamma] \in \pi_1(X^n, x_0)$ for some n and let $H\colon I \times I \to X$ be a null-homotopy of γ in X. Then also H has compact image, so $H(I \times I) \subseteq X^m$ for large enough $m \geq n$, which shows $[\gamma]$ is already trivial in $\pi_1(X^m, x_0)$.

Again using that for $[\gamma] \in \pi_1(X^n, x_0)$, the image $\gamma(I)$ is contained in a finite subcomplex of X^n, we conclude from Theorem 2.28(i) that $\pi_1(X^{n-1}, x_0) \cong \pi_1(X^n, x_0)$ for $n \geq 3$. Hence $\pi_1(X, x_0) \cong \operatorname{colim}_n \pi_1(X^n, x_0) \cong \pi_1(X^2, x_0)$. Lastly, $x_0 \in X^2$ being the only 0-cell in X^2 means X^2 is constructed just like a presentation complex and we conclude as in Theorem 2.30. □

Finally, let us discuss **gluings** of CW complexes and the effect on fundamental group and homology. The input of a gluing is a CW pair (X, A) and a cellular map $f\colon A \to Y$ from the subcomplex A to another CW complex Y as **gluing map**. The output of the gluing is the space Z in the pushout square

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow{\overline{f}} & Z \end{array}$$

in Top. It is a CW complex endowed with the filtration $Z^n = \overline{f}(X^n) \cup j(Y^n)$. The set of n-cells of Z consists of the n-cells of Y and those n-cells of X that do not lie in A. Composing corresponding characteristic maps with j and $\overline{f}$, respectively, gives characteristic maps for Z, and we have

$$\begin{aligned} \operatorname{colim}_n Z^n &= \operatorname{colim}_n((\overline{f} \textstyle\coprod j)(X^n \coprod Y^n)) = (\overline{f} \coprod j)(\operatorname{colim}_n(X^n \coprod Y^n)) = \\ &= (\overline{f} \textstyle\coprod j)(X \coprod Y) = Z \end{aligned}$$

because $\overline{f} \coprod j$ is an identification map. We call the above pushout a **cellular pushout**. In a cellular pushout, also $\overline{f}$ is cellular and $j\colon Y \to Z$ is the inclusion of a subcomplex by Theorem 1.42 (ii). Combining Theorems 6.14 and 2.27 shows the fundamental group functor sends cellular pushouts to pushouts of groups.

Theorem 6.19 (van Kampen—CW Version)
Let

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow{\overline{f}} & Z. \end{array}$$

be a cellular pushout with nonempty path connected CW complexes A, X, and Y. Pick $a_0 \in A$ and set $x_0 = i(a_0)$, $y_0 = f(a_0)$, $z_0 = \overline{f}(i(a_o)) = j(f(a_0))$. Then

$$\begin{array}{ccc} \pi_1(A, a_0) & \xrightarrow{\pi_1(f)} & \pi_1(Y, y_0) \\ {\scriptstyle \pi_1(i)}\downarrow & & \downarrow{\scriptstyle \pi_1(j)} \\ \pi_1(X, x_0) & \xrightarrow{\pi_1(\overline{f})} & \pi_1(Z, z_0) \end{array}$$

is a pushout in Group.

Combining Theorems 6.14 and 5.29 shows that cellular pushouts give rise to a LES in any homology theory (H_*, ∂_*).

Theorem 6.20 (Mayer–Vietoris Sequence for Cellular Pushouts)
Let

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow{\overline{f}} & Z. \end{array}$$

be a cellular pushout. Let $B \subseteq A$ and $B' \subseteq Y$ be subsets with $f(B) \subseteq B'$. Then we have a natural LES

$$\to H_n(A, B) \xrightarrow{(H_n(i),\, H_n(f))} H_n(X, B) \oplus H_n(Y, B') \xrightarrow{H_n(\overline{f}) - H_n(j)} H_n(Z, B') \to .$$

For nonempty A, we can again choose points for B and B' in the theorem to obtain the **reduced Mayer–Vietoris LES for cellular pushouts**

$$\cdots \longrightarrow \widetilde{H}_n(A) \xrightarrow{(\widetilde{H}_n(f_1),\, \widetilde{H}_n(f_2))} \widetilde{H}_n(X) \oplus \widetilde{H}_n(Y) \xrightarrow{\widetilde{H}_n(g_1)-\widetilde{H}_n(g_2)} \widetilde{H}_n(Z) \xrightarrow{\partial} \cdots .$$

We point out that the isomorphism (5.28) applies to cellular pushouts. So the two CW pairs in a cellular pushout satisfy $H_n(X, A) \cong H_n(Z, Y)$ for all n.

6.2 Cellular Homology and Euler Characteristic

In the first part of this section, we explain that ordinary homology of a CW complex can be computed by means of a chain complex. To begin, we fix a homology theory (H_*, ∂_*) with values in R-mod. Further assumptions on (H_*, ∂_*) are only in place where stated.

Definition 6.21

Let (X, A) be a relative CW complex. The **cellular chain complex** $C_*^{\mathrm{CW}}(X, A; H_*)$ associated with (H_*, ∂_*) has chain modules

$$C_n^{\mathrm{CW}}(X, A; H_*) := H_n(X^n, X^{n-1})$$

and differentials

$$\partial_n^{\mathrm{CW}} : C_n^{\mathrm{CW}}(X, A; H_*) \longrightarrow C_{n-1}^{\mathrm{CW}}(X, A; H_*)$$

given by the boundary homomorphisms

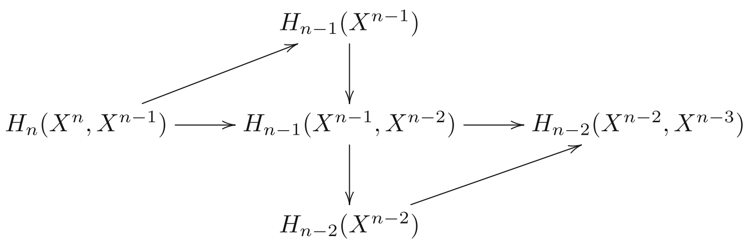
$$\partial_n : H_n(X^n, X^{n-1}) \longrightarrow H_{n-1}(X^{n-1}, X^{n-2})$$

in the LES triple sequence of the triple (X^n, X^{n-1}, X^{n-2}) from Theorem 5.2. We denote the homology of the chain complex $(C_*^{\mathrm{CW}}(X, A; H_*), \partial_*^{\mathrm{CW}})$ by:

$$H_*^{\mathrm{CW}}(X, A)$$

and call it the **cellular homology** of (X, A) associated with the theory (H_*, ∂_*).

The definition of boundary maps in the triple sequence gives a diagram



so that the composition $\partial_{n-1}^{\mathrm{CW}} \circ \partial_n^{\mathrm{CW}}$ factorizes through two arrows in a long exact sequence, so it vanishes and $(C_*(X, A; H_*), \partial_*^{\mathrm{CW}})$ is indeed a chain complex. In the case of an ordinary homology theory, it turns out that the homology of this chain complex is the original homology of the space.

Theorem 6.22
Let (X, A) be a relative CW complex and assume that (H_*, ∂_*) satisfies the dimension axiom. If (X, A) is infinite, we assume in addition that (H_*, ∂_*) is additive. Then there are isomorphisms

$$H_n(X, A) \xrightarrow{\cong} H_n^{\mathrm{CW}}(X, A)$$

for all $n \geq 0$ and these are natural with respect to cellular maps.

Proof Choose pushouts

$$\begin{array}{ccc} \coprod_{i \in I_n} S^{n-1} & \xrightarrow{\coprod q_i^n} & X^{n-1} \\ \downarrow & & \downarrow j_n \\ \coprod_{i \in I_n} D^n & \xrightarrow{\coprod Q_i^n} & X^n. \end{array}$$

Since the left hand vertical map is a cofibration, the isomorphism in (5.28), the five lemma and additivity (if X is infinite), and Corollary 5.11 provide an isomorphism

$$H_k(X^n, X^{n-1}) \cong \bigoplus_{i \in I_n} H_{k-n}(\bullet).$$

Since we assume that (H_*, ∂_*) satisfies the dimension axiom, this means

$$H_k(X^n, X^{n-1}) \cong \begin{cases} \bigoplus_{i \in I_n} H_0(\bullet) & k = n \\ 0 & k \neq n. \end{cases} \tag{6.23}$$

Considering the LES

$$H_{k+1}(X^n, X^{n-1}) \longrightarrow H_k(X^{n-1}, A) \longrightarrow H_k(X^n, A) \longrightarrow H_k(X^n, X^{n-1})$$

of the triple (X^n, X^{n-1}, A), we get an isomorphism $H_k(X^{n-1}, A) \xrightarrow{\cong} H_k(X^n, A)$ whenever $k \notin \{n, n-1\}$ according to (6.23). Thus $k > n$ implies

$$H_k(X^n, A) \cong H_k(X^{n-1}, A) \cong \cdots \cong H_k(X^{-1}, A) = H_k(A, A) = 0. \tag{6.24}$$

For $k < n$, we get $H_k(X^n, A) \cong H_k(X^m, A)$ for every $m > n$, hence $H_k(X^n, A) \cong \operatorname{colim}_m H_k(X^m, A)$ for all $k < n$. Clearly, we have $\operatorname{colim}_m H_k(X^m, A) \cong H_k(X, A)$ if X is finite dimensional. To prove that we have such an isomorphism also in the general case, consider the **homotopy colimit** of the inclusions $j_n \colon X^{n-1} \to X^n$, a model of which is given by:

$$T(X^\bullet) = \left(\coprod_{n \geq -1} X^n \times [0, 1] \right) \Big/ (x_n, 0) \sim (j_{n+1}(x_n), 1).$$

Hence $T(X^\bullet)$ is constructed by successively gluing the mapping tori $M(j_n)$ to an infinite **mapping telescope**. The mapping telescope sits in the pushout square

$$\begin{array}{ccc} \coprod_{n\geq 0} X^n & \longrightarrow & \coprod_{n\geq 0} M(j_{2n}) \\ \downarrow & & \downarrow \\ \coprod_{n\geq 0} M(j_{2n+1}) & \longrightarrow & T(X^\bullet) \end{array}$$

and the input arrows are cofibrations as we saw in Section 2.3. By Theorem 5.29, additivity, and the homotopy equivalence $M(j_n) \simeq X^n$ relative A, we have a corresponding Mayer–Vietoris sequence

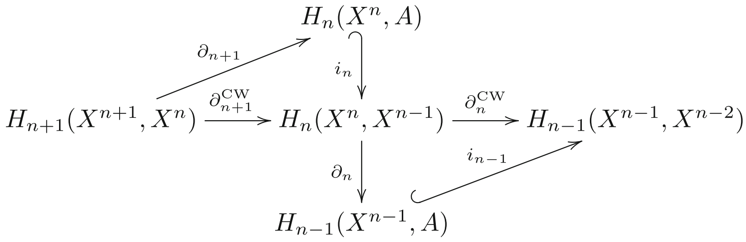
$$\bigoplus_{n\geq 0} H_k(X^n, A) \longrightarrow \bigoplus_{n \text{ odd}} H_k(X^n, A) \oplus \bigoplus_{n \text{ even}} H_k(X^n, A) \longrightarrow H_k(T(X^\bullet), A).$$

The first arrow maps the element $z_{2n} \in H_k(X^{2n}, A)$ to $(H_k(j_{2n+1})(z_{2n}), z_{2n})$ and the element $z_{2n-1} \in H_k(X^{2n-1}, A)$ to $(z_{2n-1}, H_k(j_{2n})(z_{2n-1}))$. In particular, this arrow is injective and has cokernel $H_k(T(X^\bullet), A)$. Precomposing an arrow with an automorphism leaves the cokernel unchanged, so we can precompose the first arrow with $\bigoplus_{n\geq 0}(-1)^n \operatorname{id}_{H_k(X^n,A)}$ to see that the cokernel is just the algebraic description of $\operatorname{colim}_n H_k(X^n, A)$. Hence $\operatorname{colim}_n H_k(X^n, A) \cong H_k(T(X^\bullet), A)$.

Finally, let us prove that $T(X^\bullet) \simeq X$ relative A. To see this, we endow the interval $[-1, \infty)$ with the CW structure whose 0-cells are at the integers. Then $T(X^\bullet)$ is canonically embedded as subcomplex in the product CW complex $X \times [-1, \infty)$, and we show it is a strong deformation retract. Consider the subcomplex $Y^n = T(X^\bullet) \cup (X \times [n, \infty)) \subseteq X \times [-1, \infty)$. Since (X, X^n) is a CW pair, Theorems 6.14, 2.15, and Remark 2.17 show that $X \times I$ strongly deformation retracts onto $X \times \{0\} \cup X^n \times I$. Therefore Y^n strongly deformation retracts to Y^{n+1}. Performing the strong deformation retractions successively during the time interval $[1 - 2^{-n-1}, 1 - 2^{-n-2}]$, we obtain a strong deformation retraction from $X \times [-1, \infty)$ onto $T(X^\bullet)$. This strong deformation retraction is indeed continuous because it is continuous when restricted to any skeleton of $X \times [-1, \infty)$. So $(X, A) \simeq (T(X^\bullet), A)$, hence $\operatorname{colim}_n H_k(X^n, A) \cong H_k(X, A)$ and we have proven for a general relative CW complex (X, A) and for $k < n$ that

$$H_k(X^n, A) \cong H_k(X, A). \tag{6.25}$$

Let us revisit the diagram from below Definition 6.21 with shifted degree



Since both $H_n(X^{n-1}, A)$ and $H_{n-1}(X^{n-2}, A)$ occurring in the triple sequences of (X^n, X^{n-1}, A) and (X^{n-1}, X^{n-2}, A) vanish by (6.24), the arrows i_n and i_{n-1} are injective, as indicated. It turns out that the map i_n induces an isomorphism

$$\operatorname{coker}(\partial_{n+1}) \xrightarrow{\cong} H^{\mathrm{CW}}_n(X, A) = \ker \partial^{\mathrm{CW}}_n / \operatorname{im} \partial^{\mathrm{CW}}_{n+1} \tag{6.26}$$

as we demonstrate by a diagram chase. First we have to show the induced map is well-defined. This is true because for one thing, given $z \in H_n(X^n, A)$ we have $\partial^{\mathrm{CW}}_n(i_n(z)) = i_{n-1} \circ (\partial_n \circ i_n)(z) = 0$ and for another, given $z \in \operatorname{im} \partial_{n+1}$ we have $i_n(z) \in \operatorname{im}(i_n \circ \partial_{n+1}) = \operatorname{im} \partial^{\mathrm{CW}}_{n+1}$. To see injectivity, let $z + \operatorname{im} \partial_{n+1} \in \operatorname{coker} \partial_{n+1}$ with $i_n(z) \in \operatorname{im} \partial^{\mathrm{CW}}_{n+1}$. Thus there is $z_1 \in H_{n+1}(X^{n+1}, X^n)$ with $\partial^{\mathrm{CW}}_{n+1}(z_1) = i_n(z)$. Then $i_n(\partial_{n+1}(z_1) - z) = \partial^{\mathrm{CW}}_{n+1}(z_1) - i_n(z) = i_n(z) - i_n(z) = 0$. Since i_n is injective, it follows that $z = \partial_{n+1}(z_1) \in \operatorname{im} \partial_{n+1}$, thus z represents zero in $\operatorname{coker} \partial_{n+1}$. Surjectivity follows because given $z \in \ker \partial^{\mathrm{CW}}_n$, we have $0 = \partial^{\mathrm{CW}}_n(z) = i_{n-1}(\partial_n(z))$, hence $\partial_n(z) = 0$ because i_{n-1} is injective. But $\ker \partial_n = \operatorname{im} i_n$ by exactness, so z has a preimage under i_n as desired.

The arrow ∂_{n+1} sits moreover in the long exact sequence

$$H_{n+1}(X^{n+1}, X^n) \xrightarrow{\partial_{n+1}} H_n(X^n, A) \longrightarrow H_n(X^{n+1}, A) \longrightarrow H_n(X^{n+1}, X^n)$$

of the triple (X^{n+1}, X^n, A) and the rightmost $H_n(X^{n+1}, X^n)$ vanishes by (6.23). Thus $\operatorname{coker}(\partial_{n+1}) \cong H_n(X^{n+1}, A)$, which is isomorphic to $H_n(X, A)$ by (6.25). Combining this with (6.26), we conclude $H_n(X, A) \cong H^{\mathrm{CW}}_n(X, A)$ and all isomorphisms were natural with respect to cellular maps. □

Similarly as in Corollary 5.22, we conclude that a topological finiteness condition on a space implies an algebraic finiteness condition on homology.

Corollary 6.27
Suppose that (H_*, ∂_*) *satisfies the dimension axiom and that* $H_0(\bullet)$ *is finite rank free over a principal ideal domain* R*. Let* (X, A) *be a pair of spaces, which is homotopy equivalent to a relatively finite relative CW complex. Then* $H_n(X, A)$ *is a finitely presented* R*-module for all* $n \geq 0$*.*

Proof In view of (6.23), the assumptions make sure that the cellular chain complex consists of finite rank free R-modules and subquotients of these are finitely presented because R is a principal ideal domain. □

In the special case that (X, A) is actually a CW pair, a connecting boundary homomorphism $\partial_n^{\mathrm{CW}}\colon H_n^{\mathrm{CW}}(X, A) \longrightarrow H_{n-1}^{\mathrm{CW}}(A)$ results from the following proposition by means of Theorem 4.4

Proposition 6.28
Let (X, A) *be a CW pair. Then we have a natural split SES of cellular chain complexes*

$$0 \longrightarrow C_*^{\mathrm{CW}}(A; H_*) \longrightarrow C_*^{\mathrm{CW}}(X; H_*) \longrightarrow C_*^{\mathrm{CW}}(X, A; H_*) \longrightarrow 0.$$

Proof If A is empty, there is nothing to prove. Otherwise we argue as follows. The map $i_n\colon A^n/A^{n-1} \to X^n/X^{n-1}$ is the inclusion of a subwedge of a wedge of n-spheres, so $(X^n/X^{n-1}, A^n/A^{n-1})$ is a cofibration. Sending all spheres in X^n/X^{n-1} outside A^n/A^{n-1} to the base point while leaving spheres in A^n/A^{n-1} untouched defines a retraction r_n of i_n. The canonical map from $X^n/X^{n-1} \big/ A^n/A^{n-1}$ to $X^n \cup A \,/\, X^{n-1} \cup A$ is a homeomorphism. Therefore the LES of Theorem 5.7 breaks up into split SESes

$$0 \longrightarrow \widetilde{H}_n(A^n/A^{n-1}) \longrightarrow \widetilde{H}_n(X^n/X^{n-1}) \longrightarrow \widetilde{H}_n(X^n \cup A/X^{n-1} \cup A) \longrightarrow 0.$$

Proposition 5.6 identifies these SESes with

$$0 \longrightarrow C_n^{\mathrm{CW}}(A; H_*) \longrightarrow C_n^{\mathrm{CW}}(X; H_*) \longrightarrow C_n^{\mathrm{CW}}(X, A; H_*) \longrightarrow 0$$

using that inclusions of skeleta are cofibrations. Naturality of the boundary map ∂_* in the homology theory (H_*, ∂_*) shows that these SESes assemble to a SES of chain complexes and all occurring identifications and sequences are natural with respect to cellular maps. □

It is an established abuse of notation to use the symbol "∂_n" for both chain complex differentials and connecting boundary maps in long exact homology sequences. Accordingly, also "∂_n^{CW}" has these two possible meanings by now. It should come as no surprise that the natural isomorphism $H_n(X, A) \cong H_n^{\mathrm{CW}}(X, A)$ from Theorem 6.22 identifies the boundary map ∂_n of the original homology theory (H_*, ∂_*) with the connecting boundary map ∂_n^{CW}.

Proposition 6.29
Let (X, A) be a CW pair, and assume that (H_*, ∂_*) satisfies the dimension axiom. If (X, A) is infinite, assume additionally that (H_*, ∂_*) is additive. Then we have a commutative diagram

$$\begin{array}{ccc} H_n(X,A) & \xrightarrow{\partial_n} & H_{n-1}(A) \\ \cong\downarrow & & \downarrow\cong \\ H_n^{\mathrm{CW}}(X,A) & \xrightarrow{\partial_n^{\mathrm{CW}}} & H_{n-1}^{\mathrm{CW}}(A). \end{array}$$

Proof Naturality of the boundary map ∂_n gives a commutative diagram

$$\begin{array}{ccc} H_n(X,A) & \xrightarrow{\partial_n} & H_{n-1}(A) \\ \uparrow & & \uparrow \mathrm{id} \\ H_n(X^n \cup A, A) & \xrightarrow{\partial_n} & H_{n-1}(A) \\ \downarrow & & \downarrow \\ H_n(X^n \cup A, X^{n-1} \cup A) & \xrightarrow{\partial_n} & H_{n-1}(X^{n-1} \cup A, X^{n-2} \cup A) \\ \uparrow & & \uparrow \\ H_n(X^n, X^{n-1}) & \xrightarrow{\partial_n} & H_{n-1}(X^{n-1}, X^{n-2}) \\ \uparrow & & \uparrow \\ H_n(A^n, A^{n-1}) & \xrightarrow{\partial_n} & H_{n-1}(A^{n-1}, A^{n-2}). \end{array}$$

By the snake lemma, the connecting homomorphism ∂_n^{CW} is induced by the zig zag path starting at the left middle term, going down, going right, going down, and ending at the lower right corner of the diagram. On the upper left, we can extend the diagram by the commutative square

$$\begin{array}{ccc} H_n(X^{n+1} \cup A, A) & \xrightarrow{\cong} & H_n(X,A) \\ \cong\uparrow & & \uparrow \\ \operatorname{coker} \partial_{n+1} & \longleftarrow & H_n(X^n \cup A, A) \end{array}$$

which we found as part of the proof of Theorem 6.22 and in which coker ∂_{n+1} is the cokernel of the triple sequence boundary map

$$\partial_{n+1}\colon H_{n+1}(X^{n+1} \cup A, X^n \cup A) \longrightarrow H_n(X^n \cup A, A).$$

Similarly, on the right, we can extend the diagram by

$$\begin{array}{ccc}
H_{n-1}(A) & \xleftarrow{\;\cong\;} & H_{n-1}(A^n) \\
\big\downarrow & & \big\uparrow \cong \\
H_{n-1}(X^{n-1} \cup A, X^{n-2} \cup A) & & \\
\big\uparrow & & \\
H_{n-1}(X^{n-1}, X^{n-2}) & & \operatorname{coker} \partial_n \\
\big\uparrow & & \big\uparrow \\
H_{n-1}(A^{n-1}, A^{n-2}) & \longleftarrow & H_{n-1}(A^{n-1})
\end{array}$$

where this time coker ∂_n is the cokernel of the pair sequence boundary map

$$\partial_n\colon H_n(A^n, A^{n-1}) \longrightarrow H_{n-1}(A^{n-1}).$$

This extension of the diagram commutes because it is induced by an underlying commutative diagram of pairs of spaces. Now the left extension, the zig zag path, the right extension, and the upper boundary map in the fully extended diagram induce the desired commutative square on the subquotients $H_n^{\mathrm{CW}}(X, A)$ and $H_{n-1}^{\mathrm{CW}}(A)$ of $H_n(X^n \cup A, X^{n-1} \cup A)$ and $H_{n-1}(A^{n-1}, A^{n-2})$, respectively. □

To conclude this section, we discuss what might be the most iconic concept of algebraic topology: the **Euler characteristic**. The historic background is **Euler's polyhedron formula**, which asserts for every convex polyhedron that

$$\chi = V - E + F = 2$$

where V, E, and F denote the number of vertices, edges, and faces of the polyhedron, respectively. We can now give a vast generalization of this fact: we define the Euler characteristic for any finite CW complex X, and we prove that it is an invariant of the homotopy type of X.

Definition 6.30

Let X be a finite CW complex. The **Euler characteristic** of X is the integer

$$\chi(X) = \sum_{n \geq 0} (-1)^n |\pi_0(X^n \setminus X^{n-1})|.$$

So in words, the Euler characteristic is the alternating sum over the number of cells in X. Let us define the **n-th Betti number** of a topological space X by $b_n(X) = \operatorname{rank}_{\mathbb{Z}} H_n^{\text{sing}}(X) \in \{0, 1, 2, \ldots\} \cup \{\infty\}$ where as usual, the rank of an abelian group is defined as the cardinality of a maximal linearly independent subset. So equivalently, $b_n(X) = \dim_{\mathbb{Q}}(H_n^{\text{sing}}(X) \otimes_{\mathbb{Z}} \mathbb{Q})$ (and also equivalently $b_n(X) = \dim_{\mathbb{Q}} H_n^{\text{sing}}(X; \mathbb{Q})$ because the functor $(-) \otimes_{\mathbb{Z}} \mathbb{Q}$ is left and right exact).

Theorem 6.31 (Euler–Poincaré Formula)
Let X be a finite CW complex. Then

$$\chi(X) = \sum_{n \geq 0} (-1)^n b_n(X).$$

Proof Let us simply write $C_* = C_*^{\text{CW}}(X; H_*^{\text{sing}})$ for the cellular chain complex of X. By Theorem 6.22, we have to show that

$$\sum_{n \geq 0} (-1)^n \operatorname{rank}_{\mathbb{Z}} C_n = \sum_{n \geq 0} (-1)^n \operatorname{rank}_{\mathbb{Z}} H_n(C_*). \tag{6.32}$$

But this is a general fact for finite chain complexes of finitely generated abelian groups as we will prove now. Consider the two SESes

$$0 \longrightarrow B_n \longrightarrow Z_n \longrightarrow H_n(C_*) \longrightarrow 0,$$

$$0 \longrightarrow Z_n \longrightarrow C_n \xrightarrow{\partial_n} B_{n-1} \longrightarrow 0,$$

where as before Z_n and B_n denote cycles and boundaries. Since $\operatorname{rank}_{\mathbb{Z}}$ is additive for SESes (alternatively, since $\dim_{\mathbb{Q}}$ is additive after applying $(-) \otimes_{\mathbb{Z}} \mathbb{Q}$), we obtain

$$\operatorname{rank}_{\mathbb{Z}} Z_n = \operatorname{rank}_{\mathbb{Z}} B_n + \operatorname{rank}_{\mathbb{Z}} H_n(C_*),$$

$$\operatorname{rank}_{\mathbb{Z}} C_n = \operatorname{rank}_{\mathbb{Z}} Z_n + \operatorname{rank}_{\mathbb{Z}} B_{n-1}.$$

Forming the alternating sum over the second equality, and replacing the term $\operatorname{rank}_{\mathbb{Z}} Z_n$ by means of the first inequality, the terms $\operatorname{rank}_{\mathbb{Z}} B_n$ cancel out in pairs so that we obtain the asserted equality. □

Corollary 6.33
Let X and Y be homotopy equivalent finite CW complexes. Then

$$\chi(X) = \chi(Y).$$

It can, for example, be challenging to decide whether a given topological space is contractible. If the space admits a finite CW structure, as a first test, one can choose any such finite CW structure and compute the Euler characteristic simply by counting cells. If it is different from one, the space is not contractible. From the CW structures given in Example 2.29, we see that $\chi(\Sigma_g) = 2 - 2g$ and $\chi(N_g) = 2 - g$ for the orientable and nonorientable surface of genus g. Hence orientability and Euler characteristic jointly distinguish all closed connected 2-manifolds. The CW structure of the product $X \times Y$ of two finite CW complexes X and Y given in Theorem 6.13 shows moreover that $\chi(X \times Y) = \chi(X)\chi(Y)$. In particular $\chi(\mathbb{T}^d) = 0$ and more generally $\chi(S^1 \times X) = 0$ for any finite CW complex X. Similarly, for an n-fold covering map $Y \to X$ of a finite CW complex X, we have $\chi(Y) = n \cdot \chi(X)$ in view of Exercise 6.3. This shows that a nonzero Euler characteristic obstructs nontrivial self-coverings of a space. So unless $g = 1$, a covering map $\Sigma_g \to \Sigma_g$ must be the identity map. In addition to the multiplicativity properties of the Euler characteristic for products and coverings, the Euler characteristic is also additive in the following sense.

Theorem 6.34
For a cellular pushout

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow{\bar{f}} & Z \end{array}$$

with finite CW complexes X and Y, we have $\chi(Z) = \chi(X) + \chi(Y) - \chi(A)$.

Proof We consider the Mayer–Vietoris LES in Theorem 6.20 for singular homology as a chain complex with trivial homology. Applying (6.32) to this chain complex gives

$$\sum_{n\geq 0}(-1)^n \operatorname{rank}_{\mathbb{Z}} H_n(Z) - \sum_{n\geq 0}(-1)^n (\operatorname{rank}_{\mathbb{Z}} H_n(X) + \operatorname{rank}_{\mathbb{Z}} H_n(Y)) +$$
$$+ \sum_{n\geq 0}(-1)^n \operatorname{rank}_{\mathbb{Z}} H_n(A) = 0$$

so that the theorem follows from the Euler–Poincaré formula. □

With this inclusion–exclusion principle, the Euler characteristic of a space can be computed successively by decomposing it into smaller parts.

6.3 Computing Cellular Homology

We will now explain how to make the cellular chain complex explicit in terms of degrees of self-maps of spheres induced by attaching maps. The homology of a CW complex can then be computed algebraically as we already practiced in the case of simplicial homology of Δ-complexes in Sects. 3.2 and 3.3, and again Lemma 3.9 provides an algorithmic way of doing it. For a given relative CW complex (X, A), let us choose pushouts

$$\begin{array}{ccc} \coprod_{i\in I_n} S^{n-1} & \xrightarrow{\coprod_i q_i^n} & X^{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i\in I_n} D^n & \xrightarrow{\coprod_i Q_i^n} & X^n. \end{array}$$

Definition 6.35

For $n \geq 1$, $i \in I_n$, and $j \in I_{n-1}$, we define the **incidence number** $\mathrm{inc}^n_{i,j} \in \mathbb{Z}$ of the i-th n-cell and the j-th $(n-1)$-cell of (X, A) as the degree of the composition

$$S^{n-1} \xrightarrow{q_i^n} X^{n-1} \longrightarrow X^{n-1}/(X^{n-1}\backslash e_j^{n-1}) \xrightarrow{\overline{Q_j^{n-1}}^{-1}} D^{n-1}/S^{n-2} \xrightarrow{u_{n-1}} S^{n-1}.$$

Precomposing characteristic maps with reflections, we see that incidence numbers of relative CW complexes are only well-defined up to sign. We spell out that for $n = 1$, the conventions on the homeomorphism u_0 we agreed upon below (5.9) have the effect that

$$\mathrm{inc}^1_{i,j} = \begin{cases} 1 & \text{if } q_i^1(1) = Q_j^0(D^0) \text{ and } q_i^1(-1) \neq Q_j^0(D^0), \\ -1 & \text{if } q_i^1(-1) = Q_j^0(D^0) \text{ and } q_i^1(1) \neq Q_j^0(D^0), \\ 0 & \text{otherwise.} \end{cases}$$

It is useful to gather the incidence numbers inc^n_{ij} of (X, A) in a (possibly infinite) **incidence matrix** INC_n. By Theorem 6.4 (ii), the matrix INC_n has only finitely many nonzero entries in each column and row. Let (H_*, ∂_*) be a homology theory with values in R-mod, and let ν_n be the corresponding isomorphism from (6.23) in the case $k = n$.

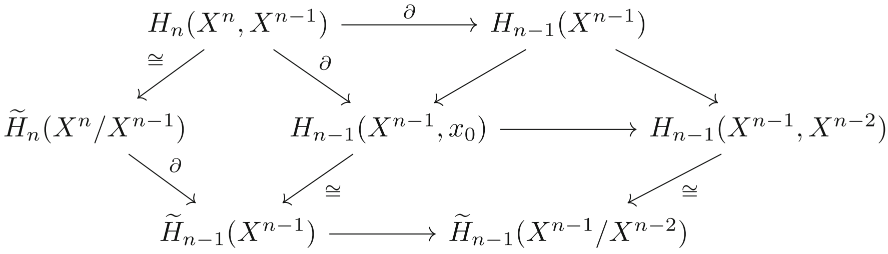
Theorem 6.36
Let (X, A) be a relative CW complex, and assume that (H_*, ∂_*) is additive if (X, A) is infinite. Choose pushouts for (X, A), and let INC_n be the associated incidence matrix. Then for all $n \geq 1$, we have a commutative diagram

$$\begin{array}{ccc}
C_n^{\mathrm{CW}}(X, A; H_*) & \xrightarrow{\partial_n^{\mathrm{CW}}} & C_{n-1}^{\mathrm{CW}}(X, A; H_*) \\
\nu_n \downarrow \cong & & \nu_{n-1} \downarrow \cong \\
\bigoplus_{i \in I_n} H_0(\bullet) & \xrightarrow{\cdot \,\mathrm{INC}_n} & \bigoplus_{i \in I_{n-1}} H_0(\bullet).
\end{array}$$

Proof The theorem is trivially true if X is empty, so for the rest of the proof, let us assume X is nonempty. We consider the following diagram in R-mod:

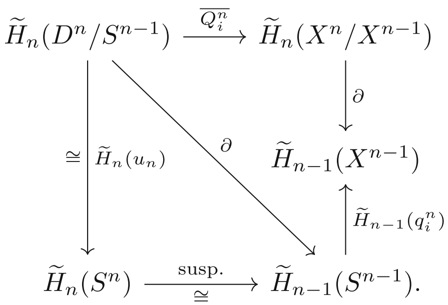
$$\begin{array}{ccccc}
H_n(X^n, X^{n-1}) & \xrightarrow{\partial} & H^{n-1}(X^{n-1}) & & \\
\cong \downarrow & & \downarrow & & \\
\widetilde{H}_n(X^n/X^{n-1}) & & H_{n-1}(X^{n-1}, X^{n-2}) & & \\
\partial \downarrow & & \downarrow \cong & & \\
\widetilde{H}_{n-1}(X^{n-1}) & \longrightarrow & \widetilde{H}_{n-1}(X^{n-1}/X^{n-2}) & \longrightarrow & \widetilde{H}_{n-1}(X^{n-1} \,/\, X^{n-1} \backslash e_j^{n-1}) \\
\uparrow \widetilde{H}_{n-1}(q_i^n) & & & & \widetilde{H}_{n-1}(\overline{Q_j^{n-1}}) \uparrow \cong \\
\widetilde{H}_{n-1}(S^{n-1}) & \xrightarrow{\cdot \mathrm{inc}_{ij}^n} & \widetilde{H}_{n-1}(S^{n-1}) & \underset{\cong}{\xleftarrow{\widetilde{H}_{n-1}(u_{n-1})}} & \widetilde{H}_{n-1}(D^{n-1}/S^{n-2}) \\
\cong \downarrow \mathrm{susp.}^{n-1} & & \cong \downarrow \mathrm{susp.}^{n-1} & & \\
\widetilde{H}_0(S^0) & & \widetilde{H}_0(S^0) & & \\
\cong \downarrow & & \cong \downarrow & & \\
H_0(\bullet) & \xrightarrow{\cdot \mathrm{inc}_{ij}^n} & H_0(\bullet) & &
\end{array}$$

The middle rectangle commutes by definition of incidence numbers and Theorem 5.33. The rectangle below commutes because every R-module homomorphism ϕ has the property $\phi(k \cdot x) = \phi(x + \cdots + x) = k \cdot \phi(x)$ for $k \in \mathbb{Z}$. To see that the upper rectangle commutes, we pick a base point $x_0 \in X^0$ and insert the auxiliary object $H_n(X^{n-1}, x_0)$ in the middle to obtain the following diagram:



The left hand square commutes because we saw in the proof of Theorem 5.7 that the LES of the cofibration (X^n, X^{n-1}) is by construction just the triple sequence of (X^n, X^{n-1}, x_0) with the given identifications. The upper triangle commutes by definition of the differential in the triple sequence. The neighboring triangle commutes because it comes from an underlying commutative diagram of pairs of spaces. Similarly, the lower parallelogram commutes because it comes from a commutative diagram of pairs of spaces before applying the identifications in (5.4) and Proposition 5.6. One quickly checks that all these commutativities imply the commutativity of the outer hexagonal diagram.

We can extend the left hand side of the first diagram with the two triangles



The upper right triangle commutes by naturality of the LESes of cofibrations as stated in Theorem 5.7. The lower left triangle commutes by definition of the suspension isomorphism $\widetilde{H}_n(S^n) \cong \widetilde{H}_{n-1}(S^{n-1})$.

Now the leftmost composition in the extended diagram, starting from $H_0(\bullet)$ at the bottom left all the way up to $H_n(X^n, X^{n-1})$ at the top left equals $\nu_n^{-1} \circ \iota_i$, where $\iota_i \colon H_0(\bullet) \to \bigoplus_{I_n} H_0(\bullet)$ is the inclusion of the i-th direct summand. Similarly, the right most composition, starting from $H_n(X^{n-1}, X^{n-2})$ in the second row at the right all the way down to $H_0(\bullet)$ at the bottom right equals $\pi_j \circ \nu_{n-1}$, where $\pi_j \colon \bigoplus_{I_{n-1}} H_0(\bullet) \to H_0(\bullet)$ is the projection to the j-th direct summand. Since the composition from $H_n(X^n, X^{n-1})$ to $H_{n-1}(X^{n-1}, X^{n-2})$ is the n-th cellular differential ∂_n^{CW}, commutativity of the diagram says that

$$\pi_j \circ \nu_{n-1} \circ \partial_n^{\mathrm{CW}} \circ \nu_n^{-1} \circ \iota_i = \cdot \mathrm{inc}_{ij}^n$$

for all $i \in I_n$ and $j \in I_{n-1}$, hence $\nu_{n-1} \circ \partial_n^{\mathrm{CW}} \circ \nu_n^{-1} = \cdot\, \mathrm{INC}_n$. □

Consequently, the product of two successive incidence matrices is zero. This is remarkable because Theorem 5.41 saying $\pi_n(S^n, \bullet) \cong \mathbb{Z}$ has an alternative homotopy theoretical proof (by the **Freudenthal suspension theorem**). Therefore an entirely homotopical definition of the notion of degree is possible from which the relation $\mathrm{INC}_n \cdot \mathrm{INC}_{n-1} = 0$ does not look quite apparent.

Example 6.37 Let us go through our examples of CW complexes in Sect. 6.1, make the corresponding cellular chain complexes explicit, and compute the homology. To do so, we fix an ordinary homology theory (H_*, ∂_*) with values in $\mathbb{Z}$-mod and coefficient module $H_0(\bullet) \cong \mathbb{Z}$; of course we think of $H_* = H_*^{\mathrm{sing}}(-; \mathbb{Z})$.

1. For the ***n*-sphere** $X = S^n$ with $n \geq 1$ and with the CW structure given above, the cellular chain complex $C_*^{\mathrm{CW}}(X, H_*)$ looks like

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0,$$

where the $\mathbb{Z}$s occur in degree 0 and n. The differentials are always zero, even if $n = 1$, by our description of incidence numbers in degree one. In fact we see that for every CW complex with only one zero cell we must have $\partial_1^{\mathrm{CW}} = 0$. It follows that

$$H_k^{\mathrm{CW}}(S^n) \cong H_k(S^n) \cong \begin{cases} \mathbb{Z} & k = 0, n \\ 0 & \text{otherwise.} \end{cases}$$

2. The chain complex $C_*^{\mathrm{CW}}(\Sigma_g, H_*)$ of the **surface of genus *g*** is given by:

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}^{2g} \longrightarrow \mathbb{Z} \longrightarrow 0.$$

So only the second differential ∂_2^{CW} is of interest. This differential is represented by the $(2g \times 1)$-matrix INC_2. To compute the entries, we recall that the 2-cell is glued in by the surface word $w = \prod_{i=1}^{g} [a_i, b_i]$ defining the pushout

$$\begin{array}{ccc} S^1 & \xrightarrow{w} & \bigvee^{2g} S^1 \\ \downarrow & & \downarrow \\ D^2 & \longrightarrow & \Sigma_g. \end{array}$$

Thus the gluing map runs through each 1-cell once in one direction and then another time in the reverse direction. So the maps in Definition 6.35 are all null-homotopic, INC_2 is the zero matrix, and the cellular chain complex has vanishing differentials only. Apparently, the homology of such a chain complex is just given by the chain modules. Thus

$$H_k(\Sigma_g) \cong \begin{cases} \mathbb{Z} & k = 0, 2, \\ \mathbb{Z}^{2g} & k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

3. The standard CW structure of **real projective d-space** $X = \mathbb{RP}^d$ given above has one cell in each dimension. Thus $C_*^{\mathrm{CW}}(X, H_*)$ is of the form

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \cdots \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow 0$$

with nonzero modules in degrees $k = 0, \dots, d$ and the incidence matrices INC_k are (1×1)-matrices. Recall that the gluing map of the k-cell is the 2-fold covering map $S^{k-1} \to \mathbb{RP}^{k-1}$, which sends pairs of antipodal points to the same point. Therefore the k-th incidence number is the degree of

$$\begin{array}{ccc} S^{k-1} & \longrightarrow & \mathbb{RP}^{k-1}/\mathbb{RP}^{k-2} \cong S^{k-1} \\ & \searrow \quad \nearrow_{\mathrm{id} \vee (-\mathrm{id})} & \\ & S^{k-1} \vee S^{k-1} & \end{array}$$

where $S^{k-1} \longrightarrow S^{k-1} \vee S^{k-1}$ collapses the equator. Because of Proposition 5.34, this gives

$$\mathrm{INC}_k = (1 + (-1)^k) = \begin{cases} 0 & k \text{ odd}, \\ 2 & k \text{ even}, \end{cases}$$

and hence the cellular chain complex looks like

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{0 \text{ or } \cdot 2} \cdots \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

depending on whether d is even or odd. It follows that

$$H_k(\mathbb{RP}^d) = \begin{cases} \mathbb{Z} & k = 0 \text{ or } (k = d \text{ and } d \text{ is odd}), \\ \mathbb{Z}/2\mathbb{Z} & 0 < k < d \text{ and } k \text{ is odd}, \\ 0 & \text{otherwise}. \end{cases}$$

4. For **complex projective d-space**, $X = \mathbb{CP}^d$, the situation is simpler. We only have cells in even degrees so $C_*^{\mathrm{CW}}(X, H_*)$ looks like

$$\longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow \cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

and this chain complex has only zero differentials. So

$$H_k(\mathbb{CP}^d) \cong \begin{cases} \mathbb{Z} & 0 \le k \le 2d \text{ and } k \text{ is even}, \\ 0 & \text{otherwise}. \end{cases}$$

6.4 Uniqueness of Ordinary Homology

In this section we will show that all ordinary and additive homology theories with a fixed coefficient module restrict to one and the same theory on CW pairs. To make this statement more precise, we define a **cellular homology theory** as in Definition 3.20 except that:

- We replace "$\mathsf{Top}^{(2)}$" with "$\mathsf{CW}^{(2)}$."
- We replace "maps" with "cellular maps," thus "homotopies" with "homotopies through cellular maps."
- Excision takes the form $H_n(A, A \cap B) \cong H_n(X, B)$ whenever X is the union of subcomplexes A and B.

Since homology theories are families of functors, a morphism of homology theories should consist of natural transformations that are compatible with the boundary transformations. This leads to the following definition that applies both to cellular and non-cellular theories.

Definition 6.38
Let (H_*, ∂_*^H) and (K_*, ∂_*^K) be two homology theories with values in R-mod. A **natural transformation of homology theories** is a family of natural transformations $\omega_*\colon H_* \to K_*$ such that for all (X, A) and all $n \in \mathbb{Z}$

$$\begin{array}{ccc} H_n(X,A) & \xrightarrow{\partial_n^H} & H_{n-1}(A) \\ \downarrow{\scriptstyle \omega_n(X,A)} & & \downarrow{\scriptstyle \omega_{n-1}(A)} \\ K_n(X,A) & \xrightarrow{\partial_n^K} & K_{n-1}(A) \end{array}$$

commutes. If each ω_n is a natural isomorphism, we call ω_* an **equivalence of homology theories**.

Homology theories and cellular homology theories form categories with morphisms given by natural transformations. Equivalences are precisely the isomorphisms in these categories. Every homology theory (H_*, ∂_*) defines a cellular homology theory $(H_* \circ F, \partial_* \circ F)$ by precomposing with the forgetful functor $F\colon \mathsf{CW}^{(2)} \longrightarrow \mathsf{Top}^{(2)}$. The cellular excision axiom is then satisfied because the CW triad $(X; A, B)$ is excisive by Theorems 6.14, 2.15, and the five lemma. With these remarks, we have the following joint reformulation of Theorem 6.22 and Proposition 6.29.

Theorem 6.39
Let (H_*, ∂_*) be an ordinary and additive homology theory with values in R-mod. Then there exists an equivalence of cellular homology theories $\omega_*\colon H_* \circ F \xrightarrow{\cong} H_*^{\mathrm{CW}}$.

Moreover, for the coefficient module, we have $\omega_0(\bullet) = \mathrm{id}_{H_0(\bullet)}$ by construction. Now we are ready to state and prove our final result.

Theorem 6.40
Suppose H_* and K_* are ordinary and additive homology theories with values in R-mod. *Let $f\colon H_0(\bullet) \to K_0(\bullet)$ be an R-homomorphism. Then there is a natural transformation*

$$\omega_*^f : H_* \circ F \longrightarrow K_* \circ F$$

of cellular homology theories with $\omega_0^f(\bullet) = f$. If f is an isomorphism, then ω_*^f is an equivalence.

Proof By Theorem 6.39, it is enough to construct a natural transformation

$$\widetilde{\omega}_*^f : H_*^{\mathrm{CW}} \longrightarrow K_*^{\mathrm{CW}}$$

with $\widetilde{\omega}_0^f(\bullet) = f$ so that $\widetilde{\omega}_*^f$ is a natural isomorphism if f is an isomorphism. To accomplish this, we will actually construct a natural transformation

$$\widetilde{\Omega}_*^f : C_*^{\mathrm{CW}}(-,-;H_*) \longrightarrow C_*^{\mathrm{CW}}(-,-;K_*)$$

of the cellular chain complex functors, which again has the properties that $\widetilde{\Omega}_0^f(\bullet) = f$ and that $\widetilde{\Omega}_*^f$ is a natural isomorphism if f is an isomorphism. Once we have done this, setting $\widetilde{\omega}_n^f(X, A) = H_n(\widetilde{\Omega}_*^f(X, A))$ does the trick because taking homology of a chain complex is functorial, as we saw at the beginning of Sect. 4.1; and because composition with a functor turns a natural transformation into a natural transformation and a natural isomorphism into a natural isomorphism. The additional naturality condition $\partial_n^{\mathrm{CW},K} \circ \widetilde{\omega}_n^f = \widetilde{\omega}_{n-1}^f \circ \partial_n^{\mathrm{CW},H}$ is just the naturality statement of Theorem 4.4 applied to the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_*^{H_*}(A) & \longrightarrow & C_*^{H_*}(X) & \longrightarrow & C_*^{H_*}(X,A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \widetilde{\Omega}_*^f(A)} & & \downarrow{\scriptstyle \widetilde{\Omega}_*^f(X)} & & \downarrow{\scriptstyle \widetilde{\Omega}_*^f(X,A)} & & \\
0 & \longrightarrow & C_*^{K_*}(A) & \longrightarrow & C_*^{K_*}(X) & \longrightarrow & C_*^{K_*}(X,A) & \longrightarrow & 0.
\end{array}$$

Now we construct $\widetilde{\Omega}_*^f$. Fix a CW pair (X, A). We view it as a relative CW complex and pick pushouts

$$\begin{array}{ccc}
\coprod_{i\in I_n} S^{n-1} & \xrightarrow{\coprod q_i^n} & X^{n-1} \\
\downarrow & & \downarrow \\
\coprod_{i\in I_n} D^n & \xrightarrow{\coprod Q_i^n} & X^n.
\end{array}$$

These determine isomorphisms $\nu_n^{H_*}$ and $\nu_n^{K_*}$ as in Theorem 6.36, and we define $\widetilde{\Omega}_*^f(X, A)$ by the composition

$$C_n^{\mathrm{CW}}(X, A; H_*) \xrightarrow{\nu_n^{H_*}} \bigoplus_{i \in I_n} H_0(\bullet) \xrightarrow{\bigoplus_i f} \bigoplus_{i \in I_n} K_0(\bullet) \xrightarrow{(\nu_n^{K_*})^{-1}} C_n^{\mathrm{CW}}(X, A; K_*).$$

This defines a chain map because we have the diagram

$$\begin{array}{ccc}
C_n^{H_*}(X,A) & \xrightarrow{c_n^{H_*}} & C_{n-1}^{H_*}(X,A) \\
\nu_n^{H_*} \downarrow \cong & & \cong \downarrow \nu_{n-1}^{H_*} \\
\bigoplus_{i\in I_n} H_0(\bullet) & \xrightarrow{\mathrm{INC}_n} & \bigoplus_{j\in I_{n-1}} H_0(\bullet) \\
\bigoplus_i f \downarrow & & \downarrow \bigoplus_j f \\
\bigoplus_{i\in I_n} K_0(\bullet) & \xrightarrow{\mathrm{INC}_n} & \bigoplus_{j\in I_{n-1}} K_0(\bullet) \\
\nu_n^{K_*} \uparrow \cong & & \cong \uparrow \nu_{n-1}^{K_*} \\
C_n^{K_*}(X,A) & \xrightarrow{c_n^{K_*}} & C_{n-1}^{K_*}(X,A).
\end{array}$$

The top and the bottom square commute by Theorem 6.36. The middle square commutes because the vertical maps are diagonal matrices in block form with the same constant entries. Clearly the chain map $\widetilde{\Omega}_*^f(X, A)$ is an isomorphism if and only if f is. Moreover, we have $\nu_0^{H_*}(\bullet) = \mathrm{id}_{H_0(\bullet)}$ and $\nu_0^{K_*}(\bullet) = \mathrm{id}_{K_0(\bullet)}$, which gives $\widetilde{\Omega}_0^f(\bullet) = f$. So the proof is complete once we show that:

(i) The chain map $\widetilde{\Omega}_*^f(X, A)$ is independent of the choice of the pushout or equivalently the choice of the characteristic maps $(Q_i^n, q_i^n)_{i \in I_n}$,

(ii) The chain map $\widetilde{\Omega}_*^f(X, A)$ is natural in (X, A) with respect to cellular maps $g\colon (X, A) \to (Y, B)$.

To address the first point, let $(P_i^n, p_i^n)_{i \in I_n}$ be another family of characteristic maps. We obtain induced homeomorphisms $\overline{P_i^n}^{-1} \circ \overline{Q_i^n}$ of D^n/S^n to itself and thus a self-map of S^n of degree ± 1 via the identification $u_n\colon D^n/S^n \xrightarrow{\cong} S^n$ from (5.9). With the suspension isomorphism $\widetilde{H}_n(S^n) \cong H_0(\bullet)$ implicit, Theorem 5.33 gives a diagram

$$\begin{array}{ccccccc}
 & & \bigoplus_{i\in I_n} H_0(\bullet) & \xrightarrow{\bigoplus_i f} & \bigoplus_{i\in I_n} K_0(\bullet) & & \\
 & \swarrow^{\bigoplus \widetilde{H}_n(\overline{Q_i^n})}_{\cong} & & & & \searrow^{\bigoplus \widetilde{K}_n(\overline{Q_i^n})}_{\cong} & \\
\widetilde{H}_n(X^n/X^{n-1}) & & \Big\downarrow \bigoplus_i \pm \mathrm{id} & & \Big\downarrow \bigoplus_i \pm \mathrm{id} & & \widetilde{K}_n(X^n/X^{n-1}) \\
 & \nwarrow^{\cong}_{\bigoplus \widetilde{H}_n(\overline{P_i^n})} & & & & \nearrow^{\cong}_{\bigoplus \widetilde{K}_n(\overline{P_i^n})} & \\
 & & \cong \bigoplus_{i\in I_n} H_0(\bullet) & \xrightarrow{\bigoplus_i f} & \bigoplus_{i\in I_n} K_0(\bullet) & &
\end{array}$$

in which the two triangles commute. But also the square commutes because Theorem 5.33 asserts that the vertical arrows are given by the same diagonal matrices with entries ± 1 and the horizontal maps are likewise identical and of constant diagonal block form. Thus the upper and the lower composition define the same morphism and $\widetilde{\Omega}_*^f(X, A)$ is independent of the chosen pushout. This shows (i).

A cellular map $g\colon (X, A) \to (Y, B)$ induces a chain map

$$C_*^{\mathrm{CW}}(X, A; H_*) \xrightarrow{C_*^{\mathrm{CW}}(g)} C_*^{\mathrm{CW}}(Y, B; H_*).$$

This map can be made explicit as a matrix of degrees as we explain next. Given an open n-cell e_i^n for $i \in I_n(X, A)$ and an open n-cell f_j^n for $j \in I_n(Y, B)$, we define

$$\mathrm{inc}_{i,j}^n(g) \in \mathbb{Z}$$

as the degree of the composition

$$D^n/S^{n-1} \xrightarrow{\overline{Q_i^n(X)}} X^n/X^{n-1} \xrightarrow{\overline{g}} Y^n/Y^{n-1} \longrightarrow Y^n/(Y^n \setminus f_j^n) \xrightarrow{\overline{Q_j^n(Y)}^{-1}} D^n/S^{n-1}$$

again using $u_n\colon D^n/S^{n-1} \xrightarrow{\cong} S^n$. Let $\mathrm{INC}_n(g)$ be the matrix with entries $\mathrm{inc}_{i,j}^n(g)$. By a similar proof as for Theorem 6.36, we see that the diagram

$$\begin{array}{ccc}
C_n^{H_*}(X, A) & \xrightarrow{C_n^{H_*}(g)} & C_n^{H_*}(Y, B) \\
\nu_n(X,A) \Big\downarrow \cong & & \cong \Big\downarrow \nu_n(Y,B) \\
\bigoplus_{i\in I_n(X,A)} H_0(\bullet) & \xrightarrow{\mathrm{INC}_n(g)} & \bigoplus_{j\in I_n(Y,B)} H_0(\bullet)
\end{array}$$

commutes. As above, we now obtain a diagram of chain maps

$$\begin{array}{ccc} C_*^{\mathrm{CW}}(X,A;H_*) & \xrightarrow{C_*^{\mathrm{CW}}(g)} & C_*^{\mathrm{CW}}(Y,B;H_*) \\ \Big\downarrow{\scriptstyle \widetilde{\Omega}_*^f(X,A)} & & \Big\downarrow{\scriptstyle \widetilde{\Omega}_*^f(Y,B)} \\ C_*^{\mathrm{CW}}(X,A;K_*) & \xrightarrow{C_*^{\mathrm{CW}}(g)} & C_*^{\mathrm{CW}}(Y,B;K_*) \end{array}$$

which shows (ii). □

Of course both Theorems 6.39 and 6.40 also apply to cellular homology theories directly: An ordinary and additive cellular homology theory can be computed from the cellular chain complex and any homomorphism of the coefficient modules of two such theories extends to a natural transformation of the theories. This natural transformation is an equivalence if and only if the homomorphism is an isomorphism.

In the proof, we have shown that any isomorphism $f\colon H_0(\bullet) \xrightarrow{\cong} K_0(\bullet)$ of R-modules gives a natural isomorphism of the corresponding cellular chain complexes $C_*^{\mathrm{CW}}(X, A; H_*) \cong C_*^{\mathrm{CW}}(X, A; K_*)$. Moreover, for every R-module M we have constructed an ordinary and additive homology theory $H_* = H_*^{\mathrm{sing}}(-, -; M)$ in Sect. 4.4. We thus proved existence and uniqueness of *the* cellular chain complex with coefficients in M, which we can thus denote with no reference to any homology theory whatsoever as $C_*^{\mathrm{CW}}(X, A; M)$. Correspondingly, we have *the* cellular homology $H_*^{\mathrm{CW}}(X, A; M)$ with coefficients in M. If it is understood that one always works in the category $\mathsf{CW}^{(2)}$, it is fair to drop the letters "CW" and just talk about "ordinary homology with coefficients in M" denoted by $H_*(X, A; M)$.

If $Z\colon\ R\text{-}\mathsf{mod} \longrightarrow \mathbb{Z}\text{-}\mathsf{mod} = \mathsf{Ab}$ denotes the forgetful functor and M is some R-module, we have natural isomorphisms $H_*(X, A; Z(M)) \cong Z(H_*(X, A; M))$ as is immediate from Theorem 6.36. So one does not lose too much when restricting attention to the case $R = \mathbb{Z}$ from the very start. This is why according to many authors homology theories always have values in Ab. This has the virtue that one can always talk about "homology groups," which sounds more familiar than "homology modules". On the other hand, allowing values in R-mod is particularly convenient when R is a field (e. g. $R = \mathbb{Z}/2\mathbb{Z}$). In that case, homology modules are vector spaces and we find ourselves in the familiar terrain of linear algebra.

6.5 How to Proceed

Now that we have reached the end of our first course on algebraic topology, it should be helpful to conclude with some final remarks and take a glimpse on more advanced material in order to have an idea about what the next steps in your topological curriculum should be.

First, let us point out that the uniqueness question for generalized homology theories is more complicated. It is not true in general that a family of isomorphisms $\{f_*\colon H_*(\bullet) \xrightarrow{\cong} K_*(\bullet)\}$ determines an equivalence $\omega_*^{f_*}\colon H_* \circ F \longrightarrow K_* \circ F$ of generalized homology theories with $\omega_n^{f_*}(\bullet) = f_n$. It is true, however, that a natural transformation $\omega_*\colon H_* \longrightarrow K_*$ of generalized additive cellular homology theories is an equivalence if (and only if) all components at the point $\omega_*(\bullet)\colon H_*(\bullet) \xrightarrow{\cong} K_*(\bullet)$ are isomorphisms [18, Bemerkung 3.54, p. 59]. To distinguish generalized homology theories from ordinary homology, a lowercase "h" is often used for the former as in $h_*(X, A)$.

Let us give an example of a generalized homology theory. The **Freudenthal suspension theorem** says that for each fixed n the sequence of homotopy groups $\pi_{n+k}(\Sigma^k X/A, \bullet)$ stabilizes for sufficiently large k and it turns out that

$$h_n(X, A) = \operatorname{colim}_k \pi_{n+k}(\Sigma^k X/A, \bullet)$$

defines a generalized cellular homology theory with values in Ab called **stable homotopy** (we do not go into the details of the boundary map). The coefficient group $h_n(\bullet) = \operatorname{colim}_k \pi_{n+k}(S^k, \bullet)$ is called the n-th **stable stem**, usually denoted by π_n^S. We proved that $\pi_0^S \cong \mathbb{Z}$ in Theorem 5.41. In contrast, π_n^S is finite for $n > 0$ by a result of J. P. Serre [22]. Whenever R is a ring with $\mathbb{Q} \subseteq R$, we thus have $\pi_k^S \otimes_{\mathbb{Z}} R = 0$ for $k \neq 0$ and $\pi_0^S \otimes_{\mathbb{Z}} R \cong R$. Whence Theorem 6.40 and the remarks below the proof say that the homology theory $\operatorname{colim}_k \pi_{n+k}(\Sigma^k X/A, \bullet) \otimes_{\mathbb{Z}} R$ is equivalent to $H_n(X, A; R)$. Up to algebraic extension problems, it is possible to compute generalized cellular homology $h_*(X, A)$ with values in R-mod from ordinary homology by means of the so-called **Atiyah–Hirzebruch spectral sequence** [3]. The "E^2-page" is given by $E^2_{p,q} = H_p(X, A; h_q(\bullet))$. Explaining what all this means lies beyond the scope of this course, an exposition of the topic can be found in [9]. But things simplify if $\mathbb{Q} \subseteq R$. In this case the spectral sequence *collapses* and the generalized homology theory h_* is equivalent to the generalized homology theory given by $\bigoplus_{p+q=n} H_p(-, -; R) \otimes_R h_q(\bullet)$ with boundary maps $\bigoplus \partial \otimes \mathrm{id}$. The equivalence is given by the **homological Chern character**

$$\mathrm{ch}_n\colon \bigoplus_{p+q=n} H_p(X, A; R) \otimes_R h_q(\bullet) \xrightarrow{\cong} h_n(X, A)$$

defined as follows. By the last remark, we have an equivalence of homology theories

$$\begin{aligned}\bigoplus_{p+q=n} H_p(X, A; R) \otimes_R h_q(\bullet) &\cong \bigoplus_{p+q=n} (\operatorname{colim}_k \pi_{p+k}(\Sigma^k X/A, \bullet) \otimes_{\mathbb{Z}} R) \otimes_R h_q(\bullet) \cong \\ &\cong \bigoplus_{p+q=n} \operatorname{colim}_k \pi_{p+k}(\Sigma^k X/A, \bullet) \otimes_{\mathbb{Z}} h_q(\bullet).\end{aligned}$$

For an element $a \otimes b \in \operatorname{colim}_k \pi_{p+k}(\Sigma^k X/A, \bullet) \otimes_{\mathbb{Z}} h_q(\bullet)$, let $f\colon (S^{p+k}, \bullet) \longrightarrow (\Sigma^k X/A, \bullet)$ with large k be a representative of a. We define $D_{p,q}(a \otimes b)$ as the image of b under the composition

$$h_q(\bullet) \xrightarrow{\text{susp.}} h_{p+q+k}(S^{p+k}, \bullet) \xrightarrow{h_{p+q+k}(f)} h_{p+q+k}(\Sigma^k X/A, \bullet) \xrightarrow{\text{desusp.}} h_n(X/A, \bullet)$$

which gives an element of $h_n(X/A, \bullet) \cong h_n(X, A)$. Then the coproduct map

$$\bigoplus_{p+q=n} \operatorname{colim}_k \pi_{p+k}(\Sigma^k X/A, \bullet) \otimes_{\mathbb{Z}} h_q(\bullet) \xrightarrow{\bigoplus_{p,q} D_{p,q}} h_n(X, A)$$

is another equivalence of cellular homology theories. This construction is due to A. Dold [5]. In this sense, one can also compute generalized homology $h_*(X, A)$ for a CW pair (X, A) if one only knows the coefficient modules $h_*(\bullet)$. But this is often the heart of the matter. For example, the group structure of the stable stems π_n^S is at the time of writing only known for $n \leq 90$ and even here, a few uncertainties must be accepted [15]. Another example of a generalized homology theory is called **bordism** where the computation of coefficient groups amounts to classifying closed smooth manifolds up to being **bordant**. Two manifolds M_1 and M_2 are called bordant if $M_1 \coprod M_2$ bounds a manifold of one dimension higher. If one takes orientations into account, this classification becomes again a difficult task, which has created a vast amount of research [23].

Homology has a dual concept called **cohomology**. The definition of a **cohomology theory** is obtained by reversing all arrows in the Eilenberg–Steenrod axioms resulting in a family of contravariant functors H^*. Correspondingly, **singular cohomology** arises from the **cochain complex** obtained by dualizing the singular chain complex. What might sound like a dull formal exercise turns out to be a fruitful idea: unlike homology, singular cohomology possesses a **multiplicative structure** that turns it into a graded ring and homology becomes a right module over that ring. Moreover, the introduction of cohomology is particularly rewarding for the study of manifolds. In fact, the condition of being *locally* Euclidean has the *global* consequence that there exists a duality isomorphism between homology and cohomology known as **Poincaré duality**. By means of a **universal coefficient theorem**, this uncovers an otherwise hidden symmetry in the homology of manifolds. Cohomology is covered in all standard textbooks on algebraic topology, including but not limited to [4, 8, 18–20, 29].

Finally, the (debatably) most important example of a **generalized cohomology theory** is called **K-theory**. For a compact Hausdorff space X, the set $\operatorname{Vect}(X)$ of isomorphism classes of finite dimensional $\mathbb{C}$-vector bundles over X becomes a commutative monoid by taking direct sums of vector bundles. The forgetful functor from abelian groups to commutative monoids has a left adjoint called **Grothendieck completion**, and we define $K^0(X)$ as the Grothendieck completion of $\operatorname{Vect}(X)$. The **reduced K-group** $\widetilde{K}^0(X)$ of a pointed space X is the kernel of the homomorphism induced by $\bullet \to X$. We obtain the **relative K-group** $K^0(X, A) = \widetilde{K}^0(X/A)$, which gives rise to negative K-groups by setting $K^{-n}(X, A) = \widetilde{K}^0(\Sigma^n X/A)$. The celebrated **Bott periodicity theorem** says that $K^{-n}(X) \cong K^{-n-2}(X)$, hence the LES of K-theory reduces to a six-term exact sequence

$$\begin{array}{ccccc} K^0(X,A) & \longrightarrow & K^0(X) & \longrightarrow & K^0(A) \\ \uparrow & & & & \downarrow \\ K^1(A) & \longleftarrow & K^1(X) & \longleftarrow & K^1(X,A). \end{array}$$

K-theory is a powerful tool that was successfully applied to several classical problems in topology. For example, it was used to answer the beautiful question on the maximal number of linearly independent vector fields on spheres [1]. The standard reference for an introduction to topological K-theory is [2].

Exercises

6.1 Let X be a CW complex, and let $A \subseteq X$ be a subcomplex. Show that X/A is a CW complex.

6.2 Suppose X is a CW complex and the base point x_0 is a 0-cell. Show that the cone CX, the suspension SX, and the reduced suspension ΣX are CW complexes.

6.3 Let $p\colon Y \to X$ be a covering space of a CW complex X. Show that setting $Y^n = p^{-1}(X^n)$ defines the filtration of a CW structure on Y. Show that p restricts to a homeomorphism on each open cell.

6.4 Show that there is a covering map $p\colon \Sigma_g \to \Sigma_h$ if and only if $g = n(h-1)+1$ for some positive integer n. *Hint: Euler characteristic.*

6.5 Let X be a connected CW complex, and suppose that X has at least two different 0-cells $x_1, x_2 \in X^0$. Compute the ordinary homology of $Y = X/\{x_1, x_2\}$ in terms of the homology of X.

6.6 Compute the ordinary homology of the Klein bottle from the cellular chain complex.

6.7 Find the left adjoint to the functor $\mathsf{Set} \longrightarrow \mathsf{CW}$, which turns a set Y into the discrete CW complex X with $X = X^0 = Y$.

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The book was inadvertently published with incorrect page numbers in the 'List of Notations'. This has now been amended.

The updated original version for this book can be found at
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Quotient Topology

A

Given an equivalence relation "$\sim$" on a topological space X, the quotient topology on the set of equivalence classes $X/\sim$ is determined by requiring that the projection $p\colon X \to X/\sim$ be continuous and satisfy the universal property

$$\begin{array}{ccc} X & \xrightarrow{h} & Z \\ {\scriptstyle p}\downarrow & \nearrow_{\overline{h}} & \\ X/\sim & & \end{array},$$

that every continuous map $h\colon X \longrightarrow Z$ with $h(x_1) = h(x_2)$ whenever $x_1 \sim x_2$ descends to a unique continuous map $\overline{h}\colon X/\sim \longrightarrow Z$ so that $h = \overline{h} \circ p$. Of course, one can recover the equivalence relation on X from the projection p because the equivalence classes in X are precisely the preimages of the points in $X/\sim$ under p. In fact, every surjective map of sets $f\colon X \longrightarrow Y$ descends to a bijection $\overline{f}\colon X/\sim \longrightarrow Y$ where $x \sim y$ if and only if $f(x) = f(y)$. This raises the problem to abstractly characterize those surjective maps $f\colon X \longrightarrow Y$ of spaces for which $\overline{f}$ is a homeomorphism, meaning that Y carries the quotient topology with respect to $\sim$. These maps are called **quotient maps** or **identification maps**, and the characterization problem is solved by the following lemma.

Lemma A.1
Let $f\colon X \to Y$ be a surjective map of topological spaces (no continuity assumption). Then the following are equivalent:

(i) The map f induces a homeomorphism $\overline{f}\colon X/\sim \xrightarrow{\cong} Y$, where $x \sim y$ if and only if $f(x) = f(y)$.

(continued)

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(ii) For all spaces Z and all maps $g: Y \to Z$ of sets, the map $g \circ f$ is continuous if and only if g is continuous.
(iii) A subset $U \subseteq Y$ is open if and only if $f^{-1}(U) \subseteq X$ is open.
(iv) A subset $U \subseteq Y$ is closed if and only if $f^{-1}(U) \subseteq X$ is closed.

Proof It is clear that (iii) and (iv) are equivalent.

(iii) $\Rightarrow$ *(i)*. The map $\overline{f}$ is bijective by construction. It is continuous by the universal property of the quotient $X/\!\sim$ because f is continuous by one direction of (iii). To see $\overline{f}$ is open, let $U \subseteq X/\!\sim$ be an open subset. Then $V = p^{-1}(U) \subset X$ is open because $p: X \to X/\!\sim$ is continuous. Moreover, we have $f^{-1}(f(V)) = V$, so $f(V) = \overline{f}(U)$ is open by the other direction of (iii).

(i) $\Rightarrow$ *(ii)*. Let $g: Y \to Z$ be a map of sets. Suppose g is continuous. By (i), the map $\overline{f}$ is a homeomorphism and in particular continuous. So $g \circ f = g \circ \overline{f} \circ p$ is a composition of continuous maps, hence continuous. If on the other hand $g \circ f$ is continuous, then $g \circ \overline{f}$ is continuous by the universal property of the quotient topology on $X/\!\sim$. Hence $g = g \circ \overline{f} \circ \overline{f}^{-1}$ is continuous as well.

(ii) $\Rightarrow$ *(iii)*. Let τ_Y be the given topology on Y. It is clear that (iii) defines another topology τ_q on Y (called the *final* topology with respect to f). We assume that Y endowed with τ_Y satisfies (ii). But since we already proved (iii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (ii), we know that (ii) also holds true if Y is endowed with the topology τ_q. Applying (ii) to the continuous maps $g = \mathrm{id}_{(Y,\tau_Y)}$ and $g = \mathrm{id}_{(Y,\tau_q)}$, we thus see that f is continuous with respect to both topologies on Y. But then we can apply (ii) in the other direction. Since the composition $\mathrm{id}_Y \circ f = f$ is continuous no matter if we consider id_Y as a map $(Y, \tau_Y) \to (Y, \tau_q)$ or as a map $(Y, \tau_q) \to (Y, \tau_Y)$, we conclude that both these maps are continuous. Hence $\tau_Y = \tau_q$, which proves (iii). □

As a word of warning, identification maps need neither be open nor closed. For example, take $f: [0, 3) \to S^1$ with $f(t) = \exp(i\pi t)$. Then the open subset $[0, 1)$ and the closed subset $[2, 3)$ of $[0, 3)$ have the same image under f which is neither open nor closed. By definition, however, injective identification maps are homeomorphisms. In view of the equivalent characterizations in Lemma A.1, this observation provides an elegant strategy for showing that a map is a homeomorphism. Another convenient fact is that the property of being an identification map is preserved under products with locally compact spaces as the following Proposition reveals.

Proposition A.2
Let X, Y, and K be spaces and suppose K is locally compact.

(i) If X is compact, then the projection $p_Y: X \times Y \to Y$ is closed.
(ii) If $f: X \to Y$ is an identification map, then so is $f \times \mathrm{id}_K: X \times K \to Y \times K$.

Proof *Part (i).* Let $C \subseteq X \times Y$ be a closed subset. We need to show that $Y \setminus p_Y(C)$ is open. So let $y \in Y \setminus p_Y(C)$. Then for all $x \in X$, the pair (x, y) does not lie in C. Since C is closed, we conclude that each $x \in X$ comes with open neighborhoods $U_x \subseteq X$ of x and $V_x \subseteq Y$ of y such that $(U_x \times V_x) \cap C = \emptyset$. Since X is compact, there are finitely many points $x_1, \ldots, x_k \in X$ such that $\bigcup_{i=1}^k U_{x_i} = X$. Set $V = \bigcap_{i=1}^k V_{x_i}$. Then $(X \times V) \cap C = \emptyset$, so $V \subseteq Y \setminus p_Y(C)$, which shows that $y \in V$ is an inner point of $Y \setminus p_Y(C)$.

Part (ii). Let $f : X \to Y$ be an identification map. We use the characterization in Lemma A.1 (ii) to show that $f \times \mathrm{id}_K$ is an identification map, too. So let $g : Y \times K \to Z$ be a map of sets and set $h = g \circ (f \times \mathrm{id}_K)$. Assuming h is continuous, we need to show that g is continuous, the converse being trivial. Consider an open neighborhood $U \subseteq Z$ of an image point $g(y_0, k_0) \in U$. Since f is surjective, we find $x_0 \in X$ with $f(x_0) = y_0$. Therefore $h(x_0, k_0) \in U$ and by continuity of h and local compactness of K, there exists a compact neighborhood $N \subseteq K$ of k_0 such that $h(\{x_0\} \times N) \subseteq U$. Set

$$A = \{y \in Y : g(\{y\} \times N) \subseteq U\}.$$

Since $y_0 \in A$, it remains to show that $A \subseteq Y$ is open. Since f is an identification map, this is equivalent to $f^{-1}(A) \subseteq X$ being open. But

$$f^{-1}(A) = \{x \in X : h(\{x\} \times N) \subseteq U\},$$

so for the complement, we have

$$X \setminus f^{-1}(A) = p_X(h^{-1}(Z \setminus U) \cap (X \times N)),$$

which is closed because the projection $p_X : X \times N \to X$ is closed by (i). □

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List of Notation

$(-)_{\text{ab}}$	Abelianization of a group, p. 5
$[G, G]$	Derived subgroup, p. 5
$[v_0, \ldots, v_n]$	Convex hull of $v_0, \ldots, v_n$, p. 59
Ab	Category of abelian groups, p. 3
$\mathcal{C}$	Category, p. 2
$\mathcal{C}^{\text{op}}$	Opposite category, p. 4
$\chi(X)$	Euler characteristic of X, p. 152
$\operatorname{colim} D$	Universal cocone of the diagram D, p. 19
$\mathsf{Cov}_{(X,x_0)}$	Category of covering spaces over (X, x_0), p. 8
CW	Category of CW complexes, p. 141
$\deg f$	Degree of the map $f\colon S^n \longrightarrow S^n$, p. 120
Δ^n	Standard n-simplex, p. 59
$\eta\colon \mathcal{F} \to \mathcal{G}$	Natural transformation from $\mathcal{F}$ to $\mathcal{G}$, p. 6
Group	Category of groups, p. 3
$\operatorname{Hom}_{\mathcal{C}}(X, Y)$	Morphisms from X to Y, p. 2
HoTop	Homotopy category, p. 3
$\mathsf{HoTop}^{(2)}$	Homotopy category of pairs of spaces, p. 72
$\mathsf{HoTop}_\bullet$	Pointed homotopy category, p. 3
Hur	Hurewicz map, p. 98
id_X	Identity morphism of the object X, p. 2
$\operatorname{inc}_{i,j}$	Incidence numbers of a CW complex, p. 155
$\int \mathrm{d}\lambda$	Lebesgue integral, p. 132
$K\text{-}\mathsf{vect}$	Category of K-vector spaces, p. 3
$\lim D$	Universal cone of the diagram D, p. 17
$\mathbb{CP}^d$	d-dimensional complex projective space, p. 140
$\mathbb{RP}^d$	d-dimensional real projective space, p. 55
$\mathbb{T}^d$	d-dimensional torus, p. 1
$\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$	Functor from the category $\mathcal{C}$ to the category $\mathcal{D}$, p. 4
$\mathcal{N}(R)$	Normal subgroup of G generated by $R \subseteq G$, p. 11
INC_n	Incidence matrix of a CW complex, p. 155
$\operatorname{ob}(\mathcal{C})$	Objects of a category $\mathcal{C}$, p. 2

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$H_n(C_*)$	n-th homology of the chain complex C_*, p. 63
$H_n^{\Delta}(X)$	n-th simplicial homology of the Δ-complex X, p. 63
I	Unit interval [0, 1], p. 3
J	Endofunctor on $\mathsf{Top}^{(2)}$ sending (X, A) to $(A, \emptyset)$, p. 72
$K^*(X)$	K-theory, p. 166
M_f	Mapping cylinder of the map f, p. 23
M_{f_1,f_2}	Double mapping cylinder of the maps f_1 and f_2, p. 50
N_g	Nonorientable surface of genus g, p. 55
Q_i^n	Characteristic map of the i-th n-cell, p. 136
q_i^n	Attaching map of the i-th n-cell, p. 136
SX	Suspension of the space X, p. 108
$T(X^\bullet)$	Skeleton mapping telescope of the CW complex X, p. 148
V^*	Dual vector space of V, p. 5
X/A	Space obtained from X by collapsing A, p. 22
$X/\sim$	Quotient space, p. 169
X^K	Space of maps from K to X, p. 29
X^n	n-skeleton of a CW complex, p. 136
$X^{[r]}$	r-th iterated barycentric subdivision of X, p. 75
X_G	Presentation complex of the group G, p. 55
$X \amalg Y$	Coproduct of objects X and Y, p. 19
$X \vee Y$	One point union of X and Y, p. 19

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